CS 39600: Theory of Metric Embeddings **Homework 3**
Date: Feb 16, 2006 **Due:** Feb 28, 2006

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1. Let x and y denote two random nodes of the unit sphere S^{n-1}. Estimate the expected Euclidean distance between x and y.

2. Derive the following from the Measure Concentration Theorem for the sphere. If $A \subseteq S^{n-1}$ satisfies $\Pr[A] \geq \alpha$ for some constant $0 < \alpha \leq \frac{1}{2}$, then $1 - \Pr[A_t] \leq 2e^{-(t-t_0)^2 n/2}$, where t_0 is such that $2e^{-t_0^2 n/2} < \alpha$. [A_t denotes the t neighborhood of A on the unit sphere S^{n-1}, $A_t = \{x \in S^{n-1} | \exists y \in A, \|x - y\|_2 \leq t\}$]

   Comment:
   There was a misunderstanding concerning the constraint on t_0. You have to show that if t_0 fulfills the constraint then the probability of A_t is as specified. If you choose a t_0 that does not satisfy the constraint it may still be possible that the probability on A_t fulfills the bound. For example if we choose $\alpha = 0$ then the measure concencntration result tells us that we can choose $t_0 = 0$. However the above lemma would be weaker in the sense that it would require us to choose a $t_0 > 0$.

3. Use the result from above to show that the existence of an (ϵ, δ, ℓ)-cover for a node x implies the existence of a $(\sigma - 2\ell\gamma, 1 - e^{-t^2/2}, \ell)$-cover for $\gamma \geq \sqrt{2\log(2/\delta)} + t$.

   Recall that a set C is a (σ, δ, ℓ)-cover of a node x if

   $$\mathbf{Pr}_{u \in S^{n-1}} \left[\exists x' \in C : \|x - x'\| \leq \ell \text{ and } \langle x - x', u \rangle \geq \frac{\sigma}{\sqrt{n}} \right] \geq \delta$$

4. Show that any tree metric embeds isometrically into ℓ_1.

5. Show that any tree metric on n vertices embeds isometrically into $\ell_\infty^{O(\log n)}$.

   Hints:

   - Show that in any tree there exists a vertex v such that removing v from the tree, shatters the tree into components of size at most $\lfloor n/2 \rfloor$.
   - Consider trees $T_1, T_2, \ldots$ that share a common vertex v, and assume that each T_i embeds isometrically into ℓ_∞^k. Show that there is an embedding $f : \cup_i T_i \longrightarrow \mathbb{R}^k$ such that no distance expands and distances between node pairs from a T_i are preserved, i.e., for x_i,$y_i \in T_i$: $d(x_i, y_i) = \|f(x_i) - f(y_i)\|_\infty$.
   - Use the above to complete the result.