**CS 39600: Theory of Metric Embeddings** **Homework 1**
**Date:** Jan 17, 2006 **Due:** Jan 26, 2006

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1. Given $(X, d_1)$ and $(Y, d_2)$ with $X \cap Y = \{z\}$, the *gate-sum* of the two metrics is the metric $(X \cup Y, d)$, where $d|_{X\times X} = d_1$, $d|_{Y\times Y} = d_2$, and $d(x, y) = d_1(x, z) + d_2(z, y)$ for $x \in X$ and $y \in Y$.
   (a) If both $d_1$ and $d_2$ are embeddable into $\ell_1$, is $d$?
   (b) If both $d_1$ and $d_2$ are embeddable into $\ell_2$, is $d$?

2. Show that Bourgain's embedding also gives
$$(X, d) \xhookrightarrow{O(\log n)} \ell_p^{O(\log^2 n)} \quad .$$
   Use ideas from the proof for the $\ell_2$-embedding. Use Hölders inequality: $\|x\|_p \cdot \|y\|_q \geq \langle x, y\rangle$, for $1/p + 1/q = 1$.

**Definition** *A metric $(X, d)$ is called a squared $\ell_2$-metric if the metric*[1] $d' = \sqrt{d}$ *is an $\ell_2$-metric. In the following we denote the set of squared $\ell_2$-metrics with $\mathcal{L}_2$.*[2]

3. Show that any $\ell_1$-metric embeds into $\mathcal{L}_2$ isometrically.

4. Let $H_\ell$ denote the hypercube of dimension $\ell$, i.e., a graph with vertex set $V(H_\ell) = \{0, 1\}^\ell$ and edge-set $E(H_\ell) = \{(x, y) \in V \times V \mid x \text{ and } y \text{ differ in exactly one bit}\}$. Let $d$ denote the shortest-path metric on $H_\ell$. Show that $d$ is an $\mathcal{L}_2$-metric. How many dimension do you need for your embedding?

5. A metric $d$ can be viewed as a function $d : X \times X$ that assigns a length to each unordered pair of points in a space $X$. We can encode this function by a point in $\mathbb{R}^k$ with $k = \binom{|X|}{2}$, i.e., any metric on $n$ vertices corresponds to a point in $\mathbb{R}^{\binom{n}{2}}$.
   (a) Show that the set of points in $\mathbb{R}^k$ that correspond to a metric forms a convex cone.
   (b) What about points corresponding to $\ell_1$-metrics, $\ell_2$-metrics, $\ell_\infty$-metrics, $\mathcal{L}_2$-metrics? For each case either prove that it forms a cone, or find a counter-example.

6. Prove that any $n$-point metric that is embeddable into $\ell_1$ with *any* number of dimensions can actually be embedded into $\ell_1^{\binom{n}{2}}$, i.e., using only $\binom{n}{2}$ dimensions. Use the result for $\ell_1$-metrics from the previous question, the fact that $\ell_1$-metrics are cut-metrics, and apply some linear algebra.

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[1] verify that $d'$ is indeed a metric.
[2] In the literature the set of squared $\ell_2$-metrics is often denoted with $\ell_2^2$.