Approximate Message Passing

Mohsen Bayati, David Donoho, Adel Javanmard
Iain Johnstone, Marc Lelarge, Arian Maleki, Andrea Montanari

Stanford University

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Statistical estimation

$$y = f(\theta; \text{noise})$$

$$\begin{aligned} \theta &\rightarrow \text{Unknown object} \\ y &\rightarrow \text{Observations} \\ f(\,\cdot\,;\text{noise}) &\rightarrow \text{Parametric model} \end{aligned}$$

Problem: Estimate θ from observations y.

Statistical estimation

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Problem: Estimate θ from observations y.

A broad convergence

- **Statistics**
  [Genomics, . . .]
- **Data mining**
  [Collaborative filtering, Predictive analytics, . . .]
- **Signal processing**
  [Compressive sampling, . . .]
- **Inverse problems**
  [Medical imaging, Seismographic imaging,. . .]

+Data, + Computation, Exploit hidden structure

How should we think about these problems?

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Optimization?

$$\text{maximize} \quad \text{Likelihood}(\theta|y) - \text{Complexity}(\theta)$$

'Separation principle'

- Modeler/statistician proposes convex cost function.
- Optimization expert proposes simple iterative algorithm.
- Run for 20 iterations and hope for the best.

How should we think about these problems?

Beyond separation?

$$y \to \hat{\theta}^1 \to \hat{\theta}^2 \to \hat{\theta}^3 \to \dots$$

- Constrained complexity per iteration
- Fixed number of iterations (say 20)
- What is minimum MSE achievable?

Outline

- A long example (algorithm + heuristics)
- A list of theorems/pointers

A long example

What type of example?

- Image processing (because they make nice figures)
- Compressed sensing (simple)

WILL NOT MENTION SPARSITY!

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An image

$\theta =$  $\in \mathbb{C}^n$

Unknown object ($n = 512^2 \approx 2.5 \cdot 10^5$)

Noiseless linear measurements

$y = A\theta = A\cdot$ 

Want to reconstruct θ

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Want to reconstruct θ

Measurement structure

$$
\begin{aligned}
A &= \widetilde{F} R \\
\widetilde{F} &= \text{subsampled Fourier matrix} \\
R &= \begin{bmatrix} +1 & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & +1 & & \\ & & & & +1 & \\ & & & & & -1 \end{bmatrix} = \text{random modulation}
\end{aligned}
$$

$\rightarrow y \in \mathbb{C}^m, \; m = 0.17\, n$

An approach popular in this community

$$y = A\theta + z\,, \qquad z \sim \mathrm{N}(0, \sigma^2 \mathrm{I}_{m\times m})$$

$$\begin{aligned} p_{\Theta|Y}(\theta|y) &\propto \exp\Big\{-\frac{1}{2\sigma^2}\|y - A\theta\|_2^2\Big\} p_{\Theta}(\theta) \\ &\propto \prod_{a=1}^{m} \exp\Big\{-\frac{1}{2\sigma^2}(y_a - A_a^{\mathsf{T}}\theta)^2\Big\} \prod_{i=1}^{n} p_{\Theta_i}(\theta_i) \end{aligned}$$

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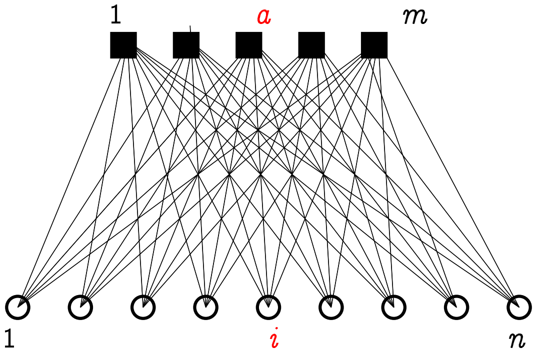
Factor graph!



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Use BP!

Many issues

- Anyone knows the prior distribution of natural images?
- Computation per iteration, memory $\Theta(mn)$.
- Very loopy graph.
- ...

Let us try something simpler!

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Let us try something simpler!

Constructing a first estimate

$$y = A\theta$$

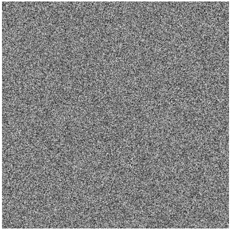
Matched filter

$$\hat{\theta}^1 = \frac{1}{m} A^\dagger y$$

How good is this?

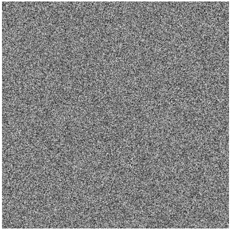
$$\mathbb{E}\,\hat{\theta}^1 = (\text{one line calculation}) = \theta$$

Check it out

$\hat{\theta}^1 = A^\dagger y =$  $\theta =$ 

Does not look that good!

Check it out

$\hat{\theta}^1 = A^\dagger y =$  $\theta =$ 

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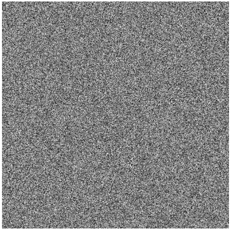
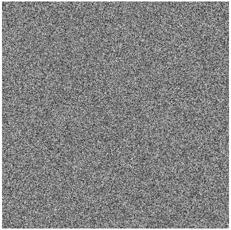
Idea



=

+ 'noise'

Idea



$=$

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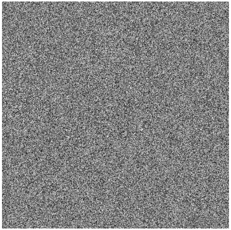


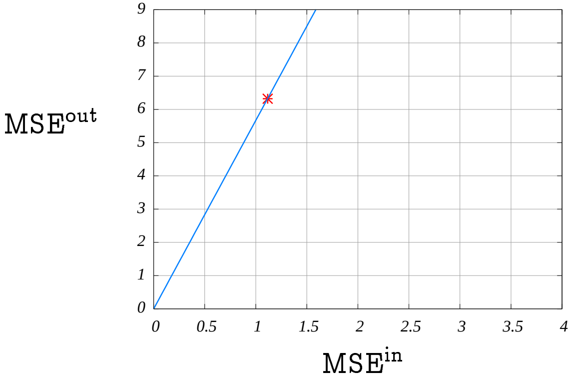
How big is the ‘noise’?

$$\mathbb{E}\{\|\hat{\theta}^1 - \theta\|_2^2\} = \text{(two lines calculation)} = \frac{1-\delta}{\delta}\,\|\theta\|_2^2$$

Matched filter blows up noise

$$\mathrm{MSE}^{\mathrm{out}} = \frac{1-\delta}{\delta}\,\mathrm{MSE}^{\mathrm{in}}$$

Let's check



Denoising

$$\hat{\theta}^1 \approx \theta + \sigma\, z\,, \qquad z_i \sim \mathrm{N}(0,1)$$

Idea: Treat $\hat{\theta}^1$ as effective observations in denoising

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Denoising by nonlocal means

$$\hat{y} = \theta + \sigma z,$$

$$\hat{\theta}_i = \frac{\sum_j W(i;j)\hat{y}_j}{\sum_j W(i;j)},$$

$$W(i;j) = \begin{cases} 1 & \text{if } \|\text{Patch}(i;\hat{y}) - \text{Patch}(j;\hat{y})\|_2^2 \le \tau\,\sigma^2, \\ 0 & \text{otherwise} \end{cases}$$

[Buades, Coll, Morel, 2005]

$\hat{\theta} \equiv \eta(\hat{y})$

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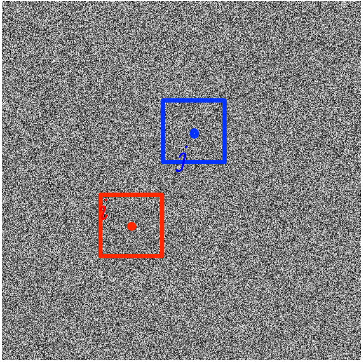
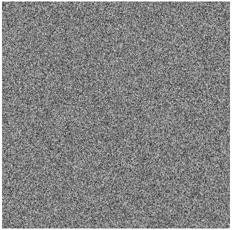
Patches



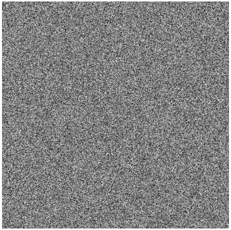
Will it work?

$$\hat{\theta}^2 = \eta(\hat{\theta}^1) = \eta(A^{\dagger}y) = \eta\Big($$



$$\Big)$$

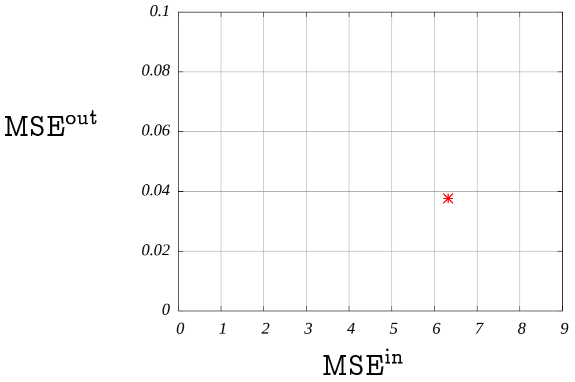
Let's try

$\hat{\theta}^1 = A^\dagger y =$ 

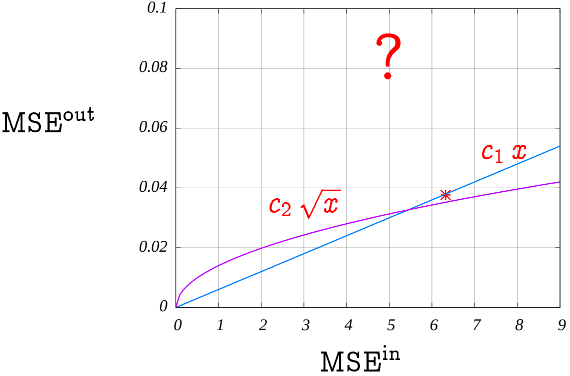
$\hat{\theta}^2 = \eta(A^\dagger y) =$ 

Better than garbage!

How much better?



How much better?



Let us repeat the denoising experiment

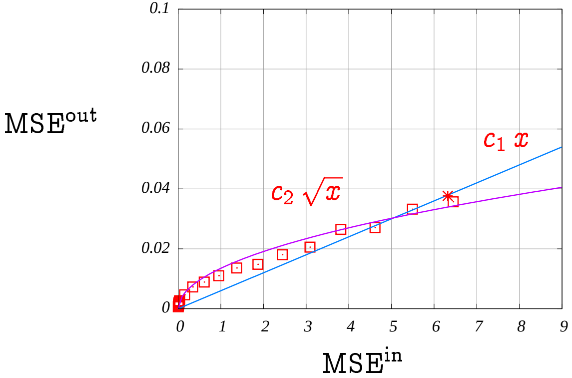
Let us repeat the denoising experiment: $y = \theta + \sigma\, z$

y

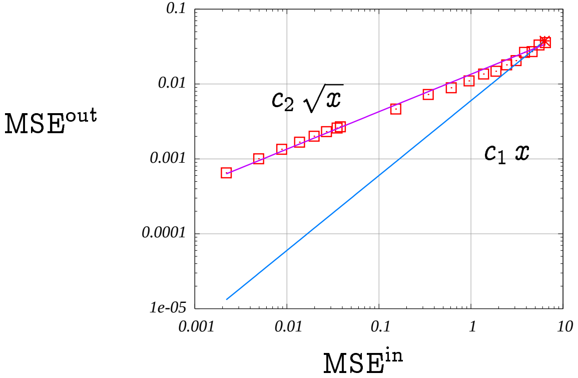
$\eta(y)$

$\sigma = 1$ $\sigma = 0.5$ $\sigma = 0.25$ $\sigma = 0.12$

Quantitatively



Quantitatively



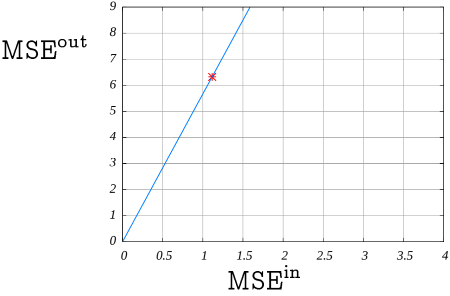
Approximate denoiser characterization

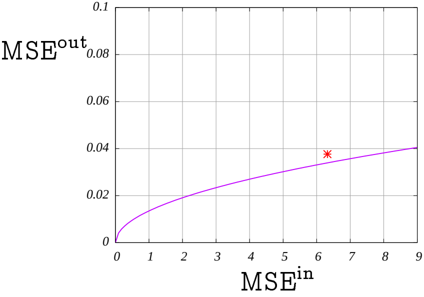
$$\mathrm{MSE}^{\mathrm{out}} = c\,\sqrt{\mathrm{MSE}^{\mathrm{in}}}$$

(enough for our purposes)

(see also Maleki, Baraniuk, Narayan, 2012)
(Arias-Castro, Willett, 2012)

What we achieved so far





Matched filter

Denoiser

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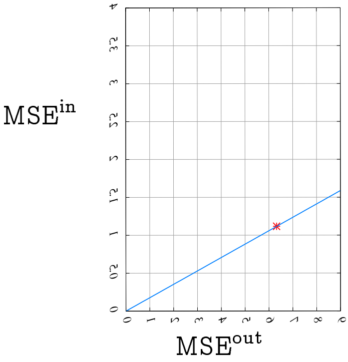

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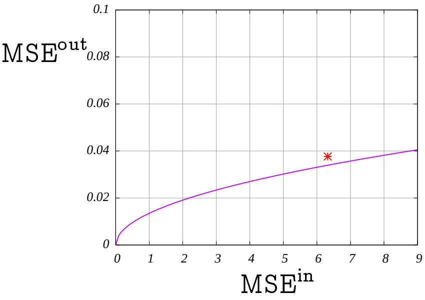

Denoiser

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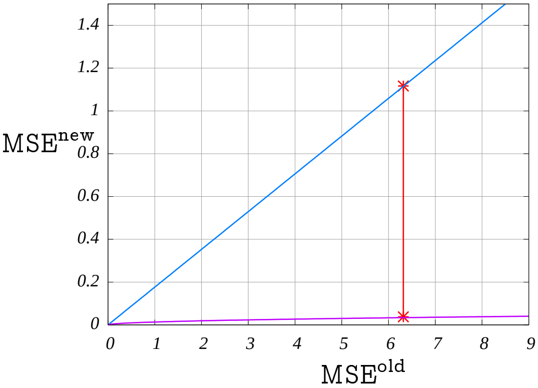

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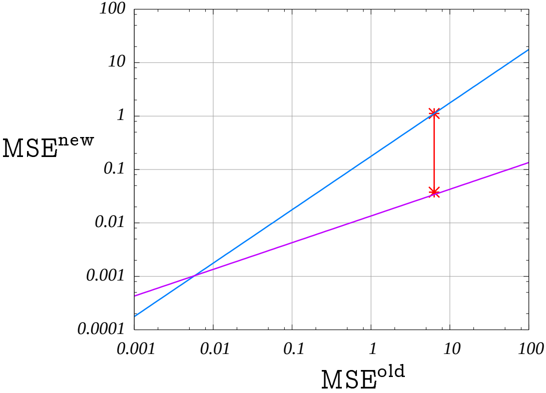
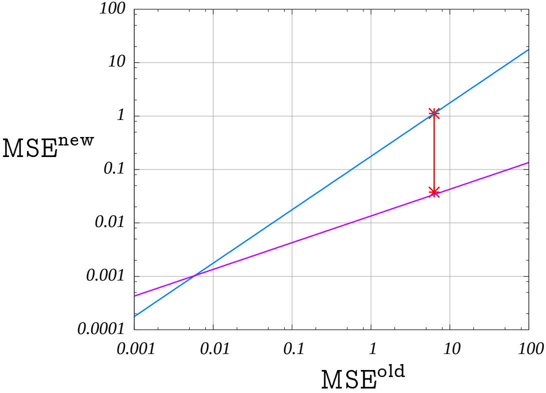


What about iterating?

What we achieved so far

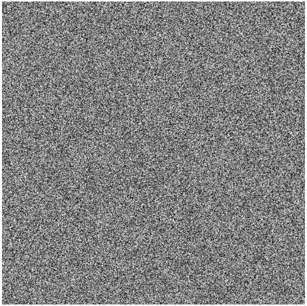


What about iterating?

How do we iterate?

Will tell you later!

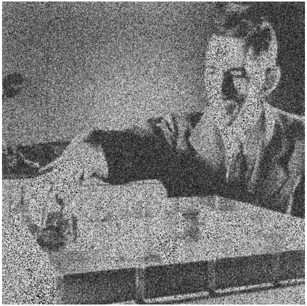
$t = 1$

$\hat{\theta}^1 =$ 

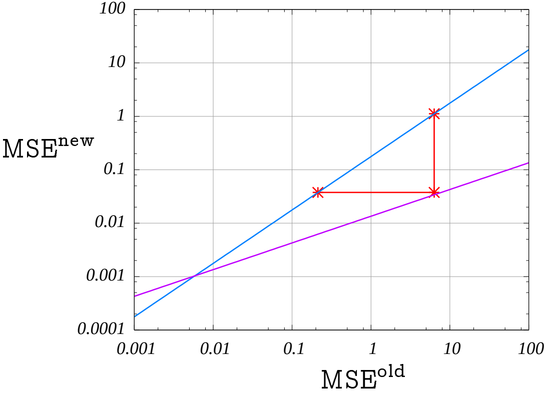
$t = 2$

$\hat{\theta}^2 =$ 

$t = 3$

$\hat{\theta}^3 =$ 

$t = 3$

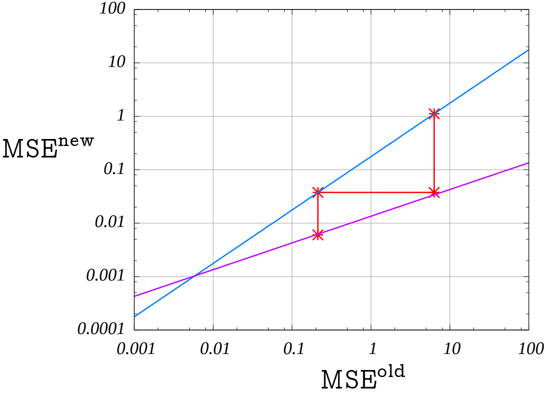


Non obvious!

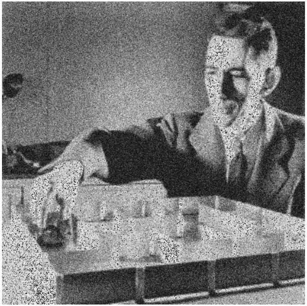
$t = 4$

$\hat{\theta}^4 =$ 

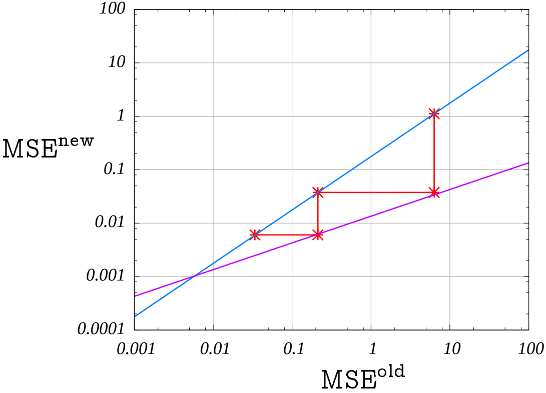
$t = 4$



$t = 5$

$\hat{\theta}^5 =$ 

$t = 5$



$t = 6$

$\hat{\theta}^6 =$ 

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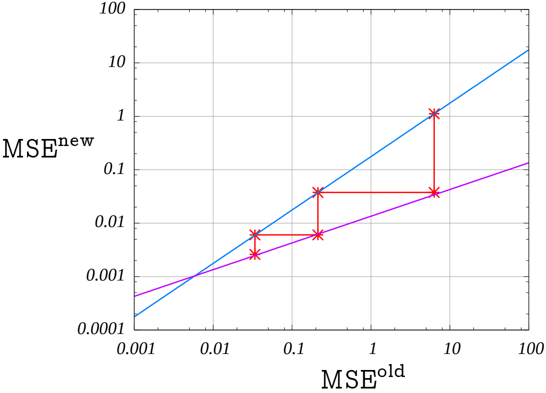
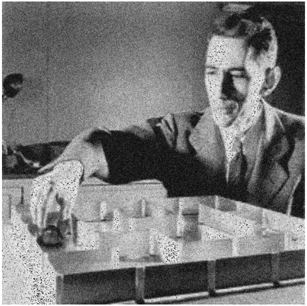
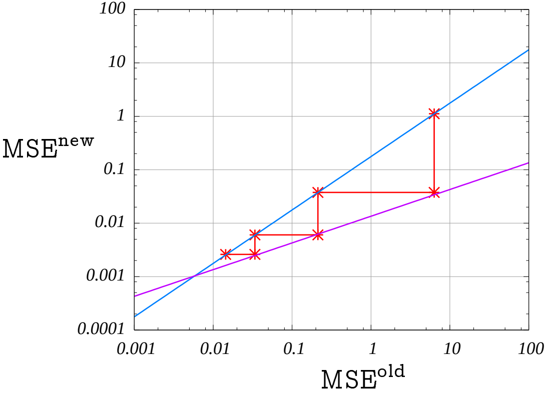


$t = 7$

$\hat{\theta}^7 =$ 

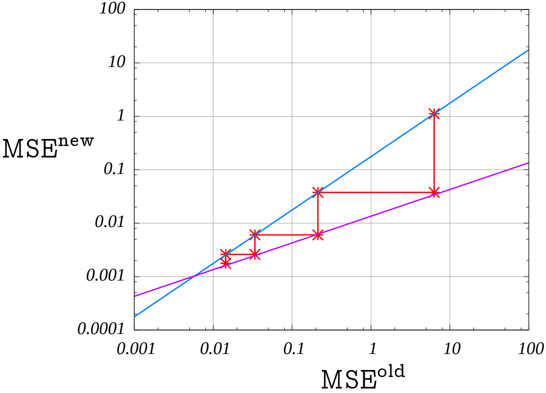
$t = 7$



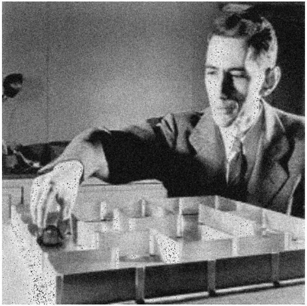
$t = 8$

$\hat{\theta}^8 =$ 

$t = 8$



$t = 9$

$\hat{\theta}^9 =$ 

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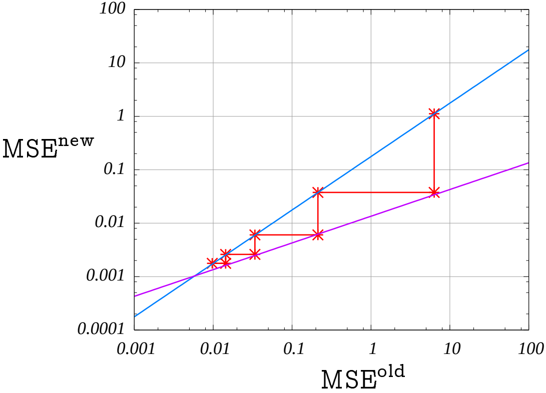

$t = 0, 1, 2, 3, \ldots, 20$

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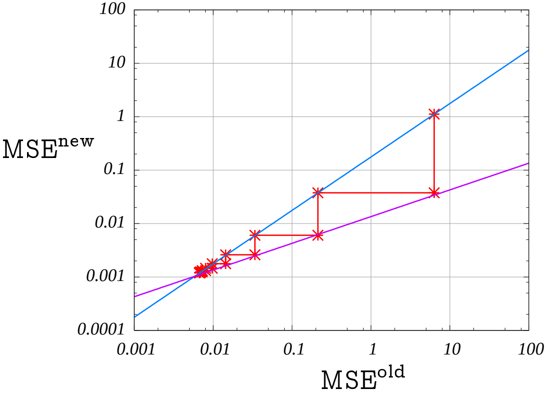

How do we iterate?

Approximate Message Passing (AMP)

$$\hat{\theta}^{2t} = \eta(\hat{\theta}^{2t-1})$$

$$\hat{\theta}^{2t+1} = \hat{\theta}^{2t} + A^{\dagger} r^t$$

$$r^t = y - A\hat{\theta}^{2t} + \mathsf{b}_t\, r^{t-1}$$

$$\mathsf{b}_t = \frac{1}{m}\text{div}\eta(\hat{\theta}^{2t-1})$$

(can be computed explicitly)

[Thouless, Anderson, Palmer, 1977, Kabashima, 2003,
Donoho, Maleki, Montanari, 2009, Donoho, Johnstone, Montanari, 2012]

Connection with Belief Propagation

$$m_{i\to j} = m_i + \varepsilon_{i\to j}$$

Linearize in $\varepsilon_{i\to j}$

Very different from naive mean field!

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Connection with Perturbation, Optimization, Statistics?

- Robustness wrt $p_X \in$DistributionClass
  ($\eta =$ minimax denoider in DistributionClass)

- Can rigorously track evolution over A random
  (What about $A = A_{\rm det} + \epsilon\, A_{\rm rand}$?)

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  (What about $A = A_{\rm det} + \epsilon\, A_{\rm rand}$?)

A list of theorems/pointers

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- Connection with optimization [Bayati, Montanari, 2011]
- Minimax theory for sparse/block-sparse/TV [Donoho, Johnstone, Montanari, 2012]
- Bayesian reconstruction up to information dimension [Donoho, Javanmard, Montanari, 2012]
- Universality [Bayati, Lelarge, Montanari, 2012]
- Analysis of Generalized AMP [Javanmard, Montanari, 2012]
- Application to sparse PCA [In preparation, 2013]

Related work

- (Non-rigorous) replica method.
  [Tanaka 2002, Guo, Verdú 2005, Kabashima, Tanaka 2009, Rangan, Fletcher, Goyal 2009, Caire, Tulino, Shamai, Verdú 2012...]

- Alternative argument for robust regression (e.g. $\min_\theta \|y - A\theta\|_1$)
  [Bean, Bickel, El Karoui, Lim, Yu 2012]

- Generalized linear models [Rangan 2011]

- Graphical model priors [Schniter et al. 2010-...]

- Low-rank matrices [Rangan, Fletcher 2012]

Conclusion

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- Can do message passing/BP without Bayesian assumptions!
- There is something between naive mean field and BP

Thanks!

Noise distribution?