Advanced Type Systems
Homework #3

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Assigned: February 1, 2006
Due: February 8, 2006

1 Soundness of the Extensional Equivalence Algorithm

Prove that the algorithm shown in class (and in Chapter 6 of ATTAPL) for deciding extensional equivalence of terms in the presence of `Unit` is *sound* with respect to definitional equivalence, *i.e.*,

$$\text{If } \Gamma \vdash e_1 : \tau \text{ and } \Gamma \vdash e_2 : \tau \text{ and } \Gamma \vdash e_1 \Leftrightarrow e_2 : \tau, \text{ then } \Gamma \vdash e_1 \equiv e_2 : \tau.$$

Hint: You may find it useful to rely on the fact that well-formed terms in this language have unique types (assuming we annotate λ-bound variables with their types).

2 Completeness in the Presence of a Top Type

Suppose we add a `Top` type to the language considered in Chapter 6. The idea is that `Top` is a supertype (in the sense of subtyping) of every other type in the language, so every well-formed term can also be given type `Top` by subsumption. For instance, if our language supported product types (which would be a straightforward extension), the `Top` type might be useful for giving a "record-polymorphic" type to the `snd` function: $\texttt{snd} : \texttt{Top} \times \tau \rightarrow \tau \stackrel{\text{def}}{=} \lambda x.\pi_2(x)$. One could then apply `snd` to any term of product type whose second component had type τ, without concern for the type of the first component (since it's guaranteed to be a subtype of `Top`).

Here are the extensions to the typing and equivalence judgments. They make use of a new subtyping judgment $\vdash \tau_1 \leq \tau_2$:

$$\frac{}{\vdash \tau \leq \tau} \qquad \frac{}{\vdash \tau \leq \texttt{Top}} \qquad \frac{\vdash \tau_2' \leq \tau_1' \quad \vdash \tau_1'' \leq \tau_2''}{\vdash \tau_1' \rightarrow \tau_1'' \leq \tau_2' \rightarrow \tau_2''}$$

$$\frac{\Gamma \vdash e : \sigma \quad \vdash \sigma \leq \tau}{\Gamma \vdash e : \tau} \qquad \frac{\Gamma \vdash e_1 : \texttt{Top} \quad \Gamma \vdash e_2 : \texttt{Top}}{\Gamma \vdash e_1 \equiv e_2 : \texttt{Top}} \qquad \frac{\Gamma \vdash e_1 \equiv e_2 : \sigma \quad \vdash \sigma \leq \tau}{\Gamma \vdash e_1 \equiv e_2 : \tau}$$

Note that all terms are equivalent when considered at type `Top`. The reason for this is simple: if all you know about e_1 and e_2 is that they both have type `Top`, then there is nothing you can do with them, so there is no way to distinguish them, and thus they are *extensionally equivalent*.

Problem: Extend the equivalence algorithm to handle `Top` and extend the logical relations proof of completeness (*i.e.*, that $\Gamma \vdash e_1 \equiv e_2 : \tau$ implies $\Gamma \vdash e_1 \Leftrightarrow e_2 : \tau$) accordingly.

Hint: `Top` is kind of like `Unit`, so most of the new cases will be trivial. One will be non-trivial.