Three Universal Relations*

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Abstract

This paper explores *universal relations*, i.e., meta-mathematical concepts that mathematicians seem to employ in all domains of mathematical reasoning. This paper presents precise mathematical definitions for three universal relations: *isomorphism* (type identity), *essential property*, and *iso-onticity*. Although no formal explanation is given as to how universal relations expedite mathematical reasoning, some intuitive arguments are presented as to why these relations, and iso-onticity in particular, seem so useful in mathematics.

1 Introduction

Consider a hypothetical automated encyclopedia of mathematics. A user of this encyclopedia wants to get information about a certain kind of mathematical object that he or she has been calling foo spaces. Since foo spaces are fairly simple it seems likely that they have already been well studied and are described somewhere in the mathematical encyclopedia. Although the user does not know the standard mathematical name for a foo space, the user can define foo spaces in terms of more basic concepts. The user states the definition of a foo space in a machine-readable language as a pair of a domain (a set) and a family of subsets of that domain such that the union of all sets in the family is the entire domain and the family of subsets is closed under arbitrary union and finite intersection. It should be clear to any human mathematician that foo spaces are the same as topological spaces and have a long history in mathematics. Unfortunately, in the automated encyclopedia a topological space is defined to be a family of sets that is closed under arbitrary union and finite intersection. From a purely formal perspective, based on these two formal definitions, a foo space is not a topological space and a topological space is not a foo space. More specifically, let F be the foo space $\langle D, X\rangle$ where D is the domain of F and X is the family of subsets of D. Under the encyclopedia's definitions, the family X is a topological space but the pair $\langle D, X\rangle$ is definitely not a topological space since it is not a family of sets. It does not matter that D can be expressed as the simple union of all sets in X; it is still the case that a pair is different from a family of sets and thus foo spaces are not topological spaces. The encyclopedia tells the user that it has no information about foo spaces.

There is clearly some inadequacy in the automated encyclopedia described above; it should have recognized a foo space as simply an alternative way of defining a topological space. The same problem arises in virtually all mathematical concepts — definitions that are really "the same" are technically quite different. As another example consider the definition of a group. A group can be defined as an algebra with one binary operation satisfying certain non-equational conditions, or it can be defined as an algebra with a binary operation, a unary operation (inverse) and a constant (the identity) satisfying certain equations. These two definitions result in technically disjoint classes of objects.

Can an automated reasoning system search for, and hopefully find, equivalences between technically distinct definitions, such as the equivalence between foo spaces and topological spaces, or the equivalence between a group as an algebra with one operation and a group as an algebra with two operations and a constant? The first step in answering this question is to find some formal characterization of when two definitions are "the same". It is tempting to try to characterize these equivalences in terms of the well known notion of isomorphism. Unfortunately, it is easy to see that there is no standard notion of isomorphism under which a foo space is isomorphic to a topological space. Two distinct topological spaces can be homeomorphic, the standard notion of isomorphism for topological spaces, but there is no

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way a topological space can be homeomorphic to an object that is not a topological space. Similarly an algebra with one operation can not be isomorphic, in the standard sense for algebras, with an algebra that has two operations and a constant.

If an automated mathematical encyclopedia can not use some standard notion of isomorphism to identify equivalences between definitions, perhaps it can use a more general category-theoretic notion of isomorphism. The basic challenge in formulating a category theoretic approach to equivalence between definitions is to define a single notion of equivalence that can be used for arbitrary mathematical definitions. The mathematical definitions that appear in textbooks and journals are not usually explicitly associated with categories. Even if a user of an automated encyclopedia explicitly associates a category with every defined concept, it is not clear that an association of a single category with each concept is sufficient for all purposes. For example, consider the two standard definitions of a lattice. A lattice can be defined as either a partially ordered set with least upper bounds and greatest lower bounds, or as an algebra satisfying certain equations. It is not clear what category should be associated with the concept of a partial order, or whether the categories associated with partial orders and with algebras are useful in recognizing the equivalence between the order-theoretic and the algebraic definitions of a lattice. As another example, consider an equivalence relation defined as a relation, i.e., a set of pairs, and defined as a partition into equivalence classes, i.e., a family of sets. Again, it is not clear what category should be associated with the set of pairs definition and what category should be associated with the equivalence class partition definition. Even if these concepts are associated with categories, it is not clear if a single association of categories with these concepts is appropriate for all possible equivalences. For many particular examples of equivalences between definitions, and perhaps even for the examples discussed so far, it is possible to associate each concept with some particular category such that some form of category theoretic equivalence is apparent. However, defining particular categories for exhibiting particular equivalences is quite different from providing a *general* definition of equivalence that can be applied to arbitrary pairs of mathematical concepts. One would like a general theory of equivalence that can be applied unambiguously to the examples given so far, and an unbounded number of other examples. In each case the general theory should determine unambiguously, at least in principle, whether or not any two given mathematical definitions are equivalent. The lack of any objective way of associating a category with arbitrary definitions, such as the definition of a partial order, partition, or equivalence relation, seems to be a major obstacle to any category-theoretic approach.

This paper takes a set-theoretic approach to the problem of recognizing equivalences between mathematical definitions. All of the mathematical definitions discussed have simple unambiguous set-theoretic formulations. For each mathematical definition, the objects which are instances of that definition, e.g. the topological spaces, are particular elements of the set-theoretic universe. Thus each definition corresponds to an unambiguous class of sets. Thus, any unambiguous notion of equivalence between classes of sets yields an unambiguous notion of equivalence between mathematical definitions.

The equivalence between the definition of a foo space and the definition of a topological space is an example of a general form of equivalence that I will call iso-onticity. The formal definition of iso-onticity given below states iso-onticity as a relationship between particular mathematical objects rather than between mathematical definitions — we say that a particular foo space is iso-ontic to a particular topological space. The concept of iso-onticity can be applied to arbitrary mathematical objects — any two mathematical objects either are or are not iso-ontic to each other. Because the concept of iso-onticity applies to arbitrary mathematical objects, iso-onticity will be called a *universal relation*, i.e., a relation that is defined on all objects. In addition to iso-onticity, we give formal definitions for two other universal relations, isomorphism and essential properties.

Intuitively, isomorphism corresponds to the notion of type identity. Consider two pennies which have just been minted by the same machine. These two pennies are physically identical for all macroscopic purposes (we might say that they are *type identical* as physical objects). Similarly we could consider two cans of diet Pepsi, or two copies of the "same" book. In each case the two objects are (for all practical purposes) physically identical. A more exact case of type identicality is given by two gaseous carbon dioxide molecules; it seems that any two such molecules in their ground energy state are truly type identical (isomorphic). In fact the notion of isomorphism seems to be universal; any object whatsoever could have a doppelganger in some parallel universe. An object is always isomorphic to its doppelganger.

It is important to distinguish isomorphism from iso-onticity. Isomorphic objects (type identical objects) are doppelgangers (like identical pennies) which are usually "materially" disjoint. On the other hand iso-ontic objects are just different "views" of "the same thing", like an equivalence relation and a partition, or the various equivalent representations of the state of a LISP processor.

The final universal relation discussed here is the notion of an *essential property*. The term essential property is *not* being used here in its standard philosophical sense. However, I have not been able to find a better term for the notion I wish to discuss. To understand this notion of essential property consider an array as a computer data structure. An array can be treated as a function. For an array A and an index value i we can let $A[i]$ denote the value of the array at the i^{th} index location. Intuitively, the value of $A[i]$ is not a property of the index i itself but rather it is a property of the array A. This situation can be contrasted with the projection function **first** defined on mathematical pairs. For any pair $\langle x, y\rangle$, $\mathbf{first}(\langle x, y\rangle)$ equals x. It seems clear that x is a property of the pair $\langle x, y\rangle$. For any pair z we say that $\mathbf{first}(z)$ is an *essential property* of z, while for an index value i, $A[i]$ need not be an essential property of the index i. A third example concerns electronic circuits. A circuit has a physical layout and a circuit topology (the topology is the way in which the components are electrically connected). The distance (in inches) between the input coupling capacitor and the output coupling capacitor is a property of the physical layout but is not a property of the circuit topology.

While the three basic universal relations discussed above seem to be based on certain natural intuitions, it is far from clear how one should go about making these notions mathematically precise. For example, it is tempting to say that y is a property of x just in case there is some function f such that $f(x)$ equals y. But this is clearly wrong because there is always a mathematical function (a set of pairs) mapping x to y.

It turns out that the notions of isomorphism, essential property, and iso-onticity can be cleanly and precisely defined in terms of a set-theoretic mathematical ontology. It seems to be empirically true that most (if not all) mathematical objects can be though of as objects in a universe of sets. Thus any relation defined over an appropriate universe of sets is in some sense universal.

2 Symmetric Set Theory

The three universal relations discussed above can be given precise definitions in terms of a set-theoretic universe with class-many ur-elements, i.e., class-many objects that are considered as distinct points without internal structure. I am not concerned with the particular first order axioms of Zermelo-Fraenkel set theory — I prefer to consider a particular "intended model" of these axioms. An intended model can be defined up to isomorphism with a few simple axioms.

A set-theoretic universe is a pair $\langle U, \in\rangle$ where U is some domain and $\in$ is a binary membership relation on U (I assume that the domain U is a set; no meta-theoretic distinction is made here between sets and classes). The relation $\in$ associates each element x of U with some particular subset of U, denoted as $\mathbf{mems}(x)$.

Definition: If x is an element of U, $\mathbf{mems}(x)$ is defined to be the subset of U given by: $\{y \text{ in } U\colon\ y \in x\}$. If $\mathbf{mems}(x)$ is empty then x will be called a *point*. For non-points x we say that x *represents* the subset $\mathbf{mems}(x)$.

To understand the significance of the axioms of symmetric set theory it is important to distinguish elements of U from the sets that those elements represent. In particular if x is a non-point *element* of U, then $\mathbf{mems}(x)$ is a *subset* of U. A particular element of U can often be thought of as representing a set of sets. For example if $\mathbf{mems}(x)$ is $\{p, z\}$ and $\mathbf{mems}(z)$ is $\{r, s\}$, then one can think of x as representing $\{p, \{r, s\}\}$. Thus the universe $\langle U, \in\rangle$ can contain representations for tuples, sets of tuples (*e.g.* relations and functions), vector spaces, and topological manifolds. The following axioms specify a particular universe of sets up to isomorphism.

Axiom One, Extensionality: There are no two distinct non-point elements x and y of U such that $\mathbf{mems}(x)$ equals $\mathbf{mems}(y)$.

Axiom Two, Strong Replacement: A subset C of U is represented by an element of U if and only if it has fewer members than U, that is, just in case $|C| < |U|$.

Axiom Three, Strong Foundation: There is no infinitely decreasing sequence of elements of U, *that is,* there is no infinite sequence $x_1, x_2, x_3, \ldots$ where $x_{i+1} \in x_i$ for all i.

Axiom Four, Infinity: There exists a *represented* infinite subset of U. Or equivalently, U must be uncountably infinite.

Axiom Five, Union: The union over any represented family of sets is represented. Equivalently, for any family F of subsets of U, if each member of F is smaller than U and the family F itself has fewer members than U, then the union of all members of F is also smaller than U.

Axiom Six, Power Set: The power set of any represented set is represented. Equivalently, for any subset C of U, if C is smaller than U then there must be more elements of U than there are subsets of C.

Axiom Seven, The Large Base Axiom: The set of all points is not represented. Equivalently, there are as many points in $\langle U, \in \rangle$ as there are elements of U.

Axiom Eight, The No Large Cardinal Axiom There does not exist any model of axioms one through seven whose domain has cardinality smaller than the cardinality of U.

Axiom eight is not necessary for the theory of universal relations but has the advantage of completely specifying the intended set theoretic universe up to isomorphism. Intuitively, the above axioms specify that the intended universe is isomorphic to the universe of all sets that can be built from a strongly inaccessible number of points (ur-elements) and that have rank less than any strongly inaccessible cardinal.

3 Three Universal Relations

Isomorphism is the universal relation most directly definable in symmetric set theory. Intuitively two objects are isomorphic just in case they have the same shape, or in other words, just in case they are the same "modulo the identity of their points." For example the set $\{p, \{p, q\}\}$ is isomorphic to the set $\{r, \{r, s\}\}$. This notion can be made precise by the following definitions.

Definition: For any element x of U, the expression $\mathbf{hmems}(x)$ will denote the set which includes x, all elements of x, all elements of elements of x, *etc*. ($\mathbf{hmems}(x)$ is the set of all things "under" x). The expression $\langle \mathbf{hmems}(x), \in \rangle$ will denote the first order structure derived by restricting the relation $\in$ to $\mathbf{hmems}(x)$.

Definition: Two objects x and y in $\langle U, \in \rangle$ are *isomorphic* just in case the sub-universes $\langle \mathbf{hmems}(x), \in \rangle$ and $\langle \mathbf{hmems}(y), \in \rangle$ are isomorphic as first order structures.

It turns out that when point-based algebraic structures and topologies are represented as sets the above universal notion of isomorphism provides the standard notions of isomorphism for these objects. For example, consider two "algebras" $\langle D, f \rangle$ and $\langle D', f' \rangle$ where D and D' are sets and f and f' are functions from D to D and D' to D' respectively. Under the standard notion of isomorphism for algebras, these two algebras are isomorphic if there exists a bijection ρ from D to D' such that for any element d of D we have that $f'(\rho(d))$ equals $\rho(f(d))$. These two algebras have standard representations as sets. For example, $\rangle D, f \langle$ is the set $\{D, \{D, f\}\}$ where D is a set and f is a set of pairs of elements of D. Provided that D and D' are sets of points, the reader can check that the algebras $\rangle D, f \langle$ and $\rangle D', f' \langle$ are isomorphic in the standard way for algebras if and only if they are isomorphic in the sense of the above definition. A similar analysis holds for the notion of homeomorphic topological spaces whose domains are sets of points.

The notion of isomorphism can be better understood in terms of the symmetries, or automorphisms, of the universe as a whole. A symmetry (automorphism) of the universe $\langle U, \in \rangle$ is a bijection ρ from U to itself such that for any two elements x and y of U, $\rho(x) \in \rho(y)$ just in case $x \in y$ (a symmetry of $\langle U, \in \rangle$ is an isomorphism of $\langle U, \in \rangle$ with itself. The next lemma completely characterizes all symmetries of any universe $\langle U, \in \rangle$ satisfying axioms one through three. The set of points in a universe $\langle U, \in \rangle$ can be thought of as a *base* for that universe upon which all other elements of U are built. It turns out that the symmetries $\langle U, \in \rangle$ exactly correspond to the permutations of the base of $\langle U, \in \rangle$ (a permutation of the base of $\langle U, \in \rangle$ is a one-to-one and onto mapping of the points in $\langle U, \in \rangle$ to themselves).

Global Symmetry Lemma: Any permutation of the points in $\langle U, \in \rangle$ has a unique extension to a full symmetry (automorphism) of $\langle U, \in \rangle$. Thus there is a natural one-to-one correspondence between the symmetries of $\langle U, \in \rangle$ and the permutations of the "base" of $\langle U, \in \rangle$.

The following lemma is fundamental for the notion of isomorphism. This lemma justifies the intuition that isomorphic objects are truly indistinguishable. (It is interesting to note that the following lemma depends on axiom seven, the large base axiom).

The Fundamental Isomorphism Lemma: Two objects x and y are isomorphic just in case there exists a symmetry (automorphism) of $\langle U, \in \rangle$ which carries x to y.

The second universal relation is the notion of essential property. To understand the notion of an essential property in the context of symmetric set theory it is useful to consider some examples. Suppose x is the set $\{p, \{p, q\}\}$ where p and q are points. We could define p as "the element of z which is also an element of an element of X." Thus p is a "definable" property of the set $\{p, \{p, q\}\}$ (the set $\{p, \{p, q\}\}$ is a standard representation for the *pair* $\langle p, q \rangle$). On the other hand consider the set of points $\{p, q, r\}$. There is "no difference" between the points p and q as elements of the set $\{p, q, r\}$. In fact p is not a definable property of the set $\{p, q, r\}$. The sense in which p is a property of $\{p, \{p, q\}\}$ but not a property of $\{p, q, r\}$ is captured by the following definition:

Definition: We say that y is *individuated* by x just in case every symmetry of $\langle U, \in \rangle$ which leaves x fixed also leaves y fixed.

Again consider the set $\{p, q, r\}$. It is easy to show that there is a symmetry of $\langle U, \in \rangle$ which moves p to q, q to r, and r to p. This symmetry leaves the set $\{p, q, r\}$ fixed while moving the point p. Thus, the point p is not individuated in the context of the set $\{p, q, r\}$. This notion of individuation seems to be related to the category theoretic notion of a "natural transformation." It is easily shown in symmetric set theory that there is no individuated linear bijection between a point-based vector space and its dual.

The notion of individuation can be further clarified by the following notion of an essential function (the notion of an essential function is somewhat related to the notion of a generic embedding between abstract data types [Dunlaing & Yap 1982]).

Definition: An *essential function* is a function f from U to U which commutes with symmetries of the universe $\langle U, \in \rangle$, *that is,* for any symmetry ρ of universe and any object x, $f\,(\rho(x))$ equals $\rho\,(f(x))$.

For any essential function f, if x is isomorphic to Y then $f(x)$ must be isomorphic to $f(y)$. In some sense an essential function is one which can be defined *purely* in terms of its arguments, i.e., the function itself does not carry information. It is easy to see that an array function A is not essential, $A[i]$ need not be isomorphic to $A[j]$ even when i and j are isomorphic. The following lemma relates essential functions and the notion of individuation: *Essential Property Lemma*:

An object y is individuated by an object x just in case there exists an *essential* function f such that y equals $f(x)$.

We can now define the notion of an essential property in the obvious way:

Definition: We say that y is an *essential property* of x just in case y is individuated by x, or equivalently, just in case there exists an essential function mapping x to y.

Essential properties can be more deeply understood by relating them to the points which objects are "made of": *Definition*: For any non-point object x, we let $\mathbf{points}(x)$ denote the set of all

points which are either elements of x, elements of elements of x, *etc.* For a point p, $\mathbf{points}(p)$ denotes the singleton set containing p.

Essential Property Point Lemma: If y is an essential property of x then $\mathbf{points}(y)$ must be a subset of $\mathbf{points}(x)$.

The above lemma says that an essential function cannot introduce points (an essential function cannot "know" which point it should introduce).

The third universal relation is iso-onticity. Two things are iso-ontic just in case each is definable in terms of the other. For example, an equivalence relation, as a set of pairs, is iso-ontic to a partition into equivalence classes, i.e., a set of sets. A Topological space defined as pair of a set and a set of subsets is iso-ontic to the same space considered as a simple set of sets.

Definition: Two objects x and y are iso-ontic just in case each is an essential property of the other.

If x and y are iso-ontic then $\mathbf{points}(x)$ must equal $\mathbf{points}(y)$. Furthermore, if x and y are iso-ontic then any essential property of x is also an essential property of y and vice versa (for example every property of an equivalence relation can be expressed as a property of the induced partition into equivalence classes).

4 Eliminating Arbitrariness in Set-Theoretic Definitions

When using set-theoretic foundations one usually takes the ordered pair of x and y to be some particular set, such as $\{x, \{x, y\}\}$. However this seems somewhat arbitrary; why not represent the pair $\langle x, y\rangle$ as $\{\{x\}, \{x, y\}\}$ or $\{y, \{y, x\}\}$? (This sort of set-theoretic arbitrariness is discussed at great length in [Benacerraf 1965].) However the set-theoretic representation of a pair is not *completely* arbitrary; the pair $\langle x, y\rangle$ could not in general be represented by the simple set $\{x, y\}$. So what is the essence of the pair $\langle x, y\rangle$ such that some set-theoretic representations "work" while others don't?

The universal relations defined in the previous section provide a way of specifying the notion of an ordered pair without making any commitment about particular set-theoretic representations. More specifically we can specify the notion of a pair by assuming the existence of three *essential* functions, **pair**, **first**, and **second** (the notion of an essential function can be easily generalized to n-ary functions). We further require that these *essential* functions satisfy the following equations for all objects x and y.

$$\begin{aligned}\mathbf{first}\,(\mathbf{pair}(x, y)) &= x \\ \mathbf{second}\,(\mathbf{pair}(x, y)) &= y\end{aligned}$$

Given that the functions **pair**, **first**, and **second** are *essential* it is possible to prove that $\mathbf{pair}(x, y)$ must be *iso-ontic* to $\{x, \{x, y\}\}$. However the set-theoretic nature of the functions need not be specified and thus one is not committed to any particular set theoretic representation for $\mathbf{pair}(x, y)$.

The essential function **pair** should be contrasted to an array A of two arguments. We have specified that **pair** be an essential function so that $\mathbf{pair}(x, y)$ is an essential property of x and y. However, a two dimensional array function A need not be essential, and in particular $A[i, j]$ can be arbitrary (it need not be an essential property of i and j).

This approach to specifying the notion of a pair is similar to modern techniques for algebraically specifying programs and data structures (*for example,* Guttag & Horning [1980], Burstall & Goguen [1977]). The major innovation of the above approach involves the semantics of the equations. Previous approaches have interpreted equational specifications over sorted algebras [Goguen *et al.* 1977]. However, sorted algebras do not provide an adequate theoretical bases for saying that x and y are essential properties of $\mathbf{pair}(x, y)$, or that $\mathbf{pair}(x, y)$ is iso-ontic to the set $\{x, \{x, y\}\}$.

5 Discussion

It is hoped that the precise mathematical theory of universal relations presented above will prove to be useful in guiding the construction of general purpose inference techniques. It seems clear that human mathematicians make use of some universal notion of isomorphism, essential property, and iso-onticity. This paper has not attempted explicate the utility of these universal concepts in determining mathematical truths. Iso-onticity is perhaps the easiest universal relation to justify on pragmatic grounds. Intuitively, it seems that one should be able to contract the space of possible mathematical concepts by collapsing any two iso-ontic concepts into a single concept. By reducing the number of concepts that can be asked about, the collapsing of iso-ontic mathematical concepts reduces the number of distinct statements that can be formulated. A reduction in the number of statements should make theorem-proving search processes more efficient. It seems likely that the concepts of isomorphism and iso-onticity can also be justified in terms of improved efficiency for automated inference.

Semantics has traditionally improved our understanding of inference systems. It is hoped that a precise theory of mathematical ontology and universal relations will shed further light on the general nature of inference.

6 References

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