26^{th} **Symposium on Computational Geometry**

Dynamic Well-Spaced Point Sets

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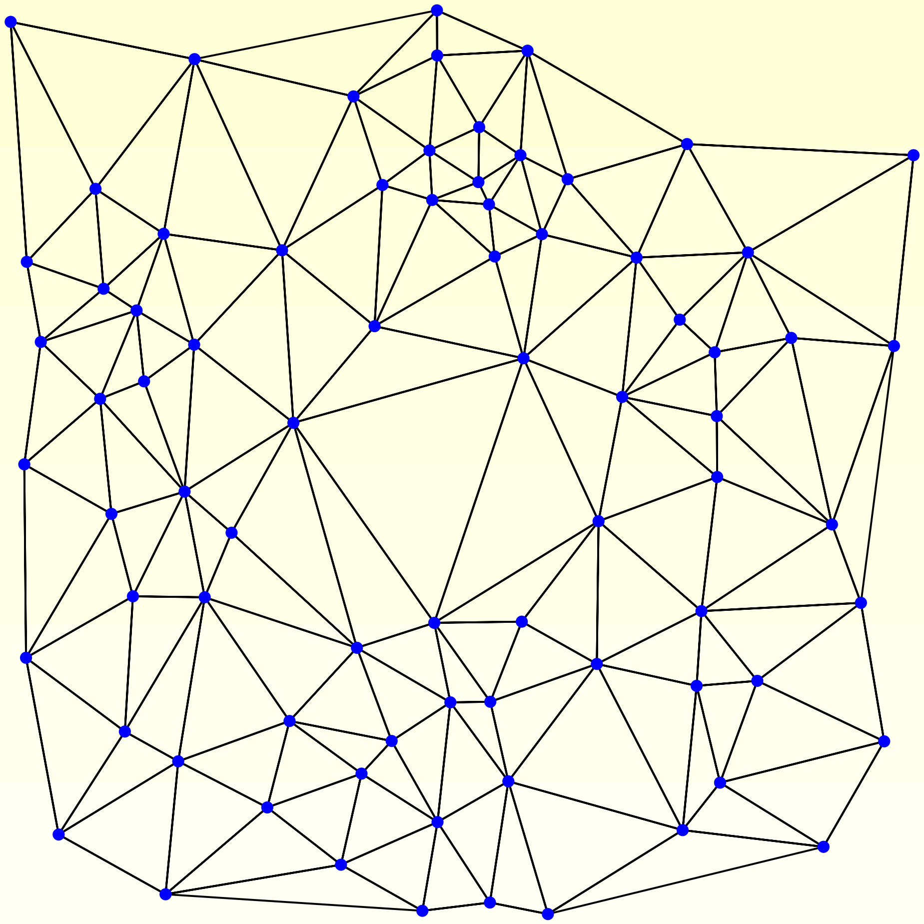
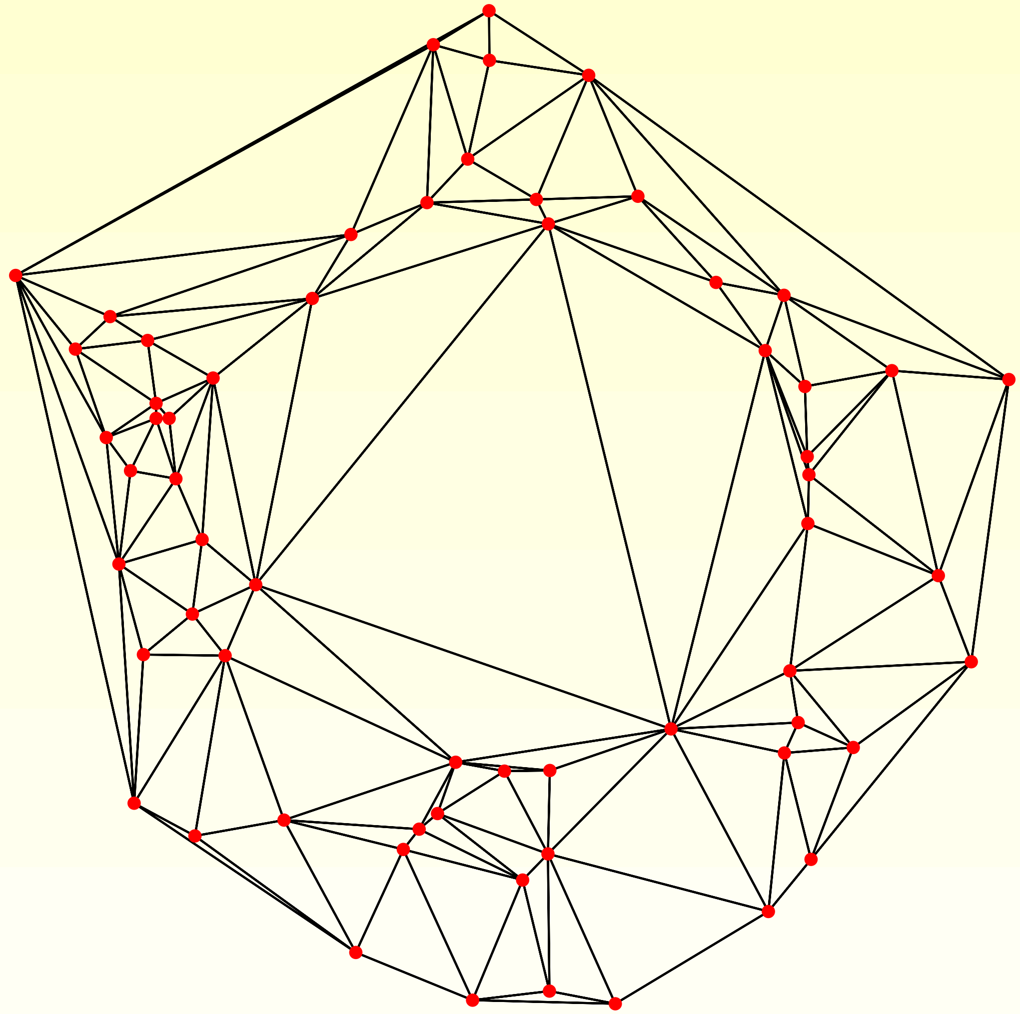
Delaunay Triangulations, 2D Examples



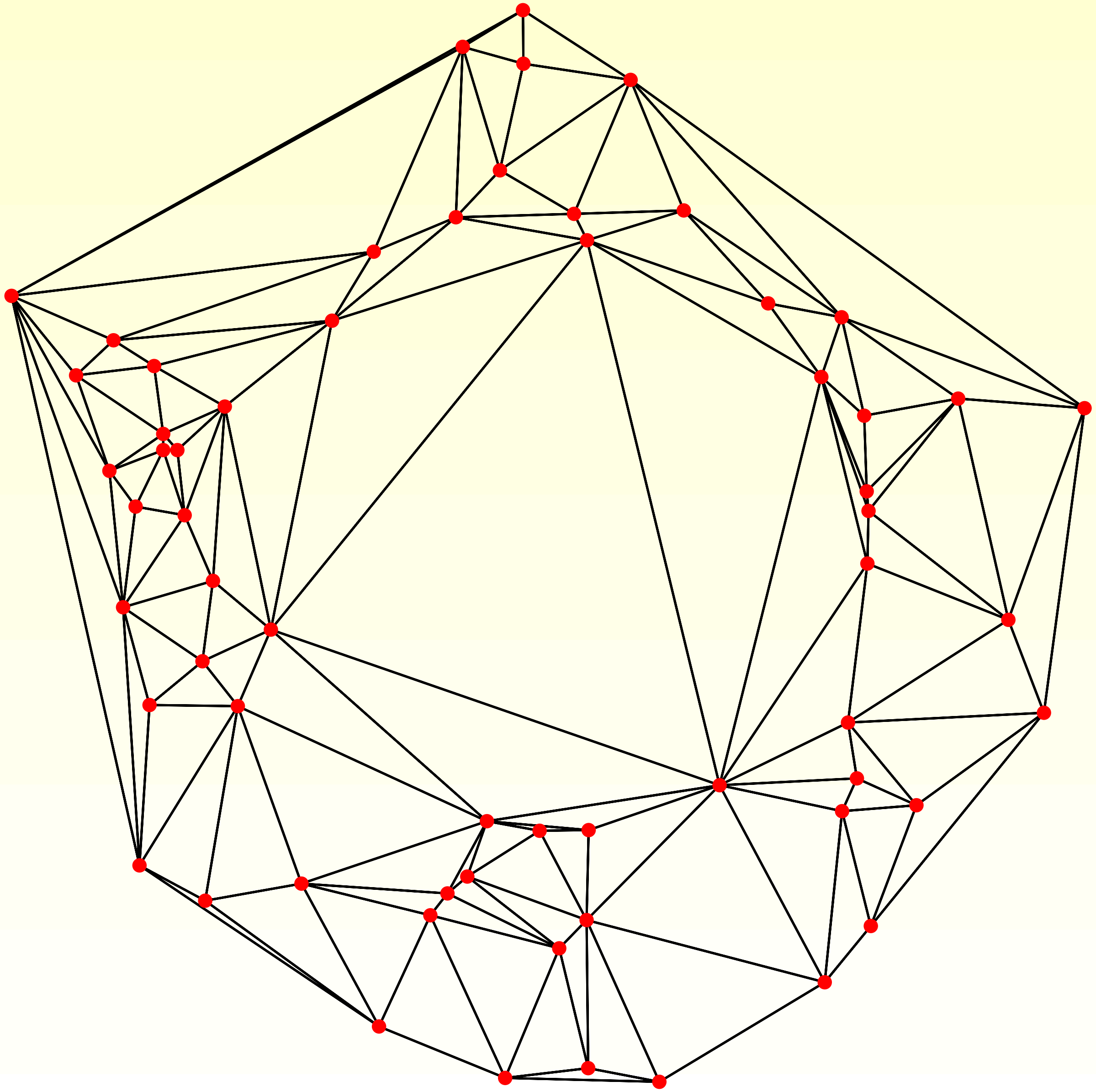
No small angles



Skinny triangles

Mesh Refinement

Input set of points does not have a good triangulation



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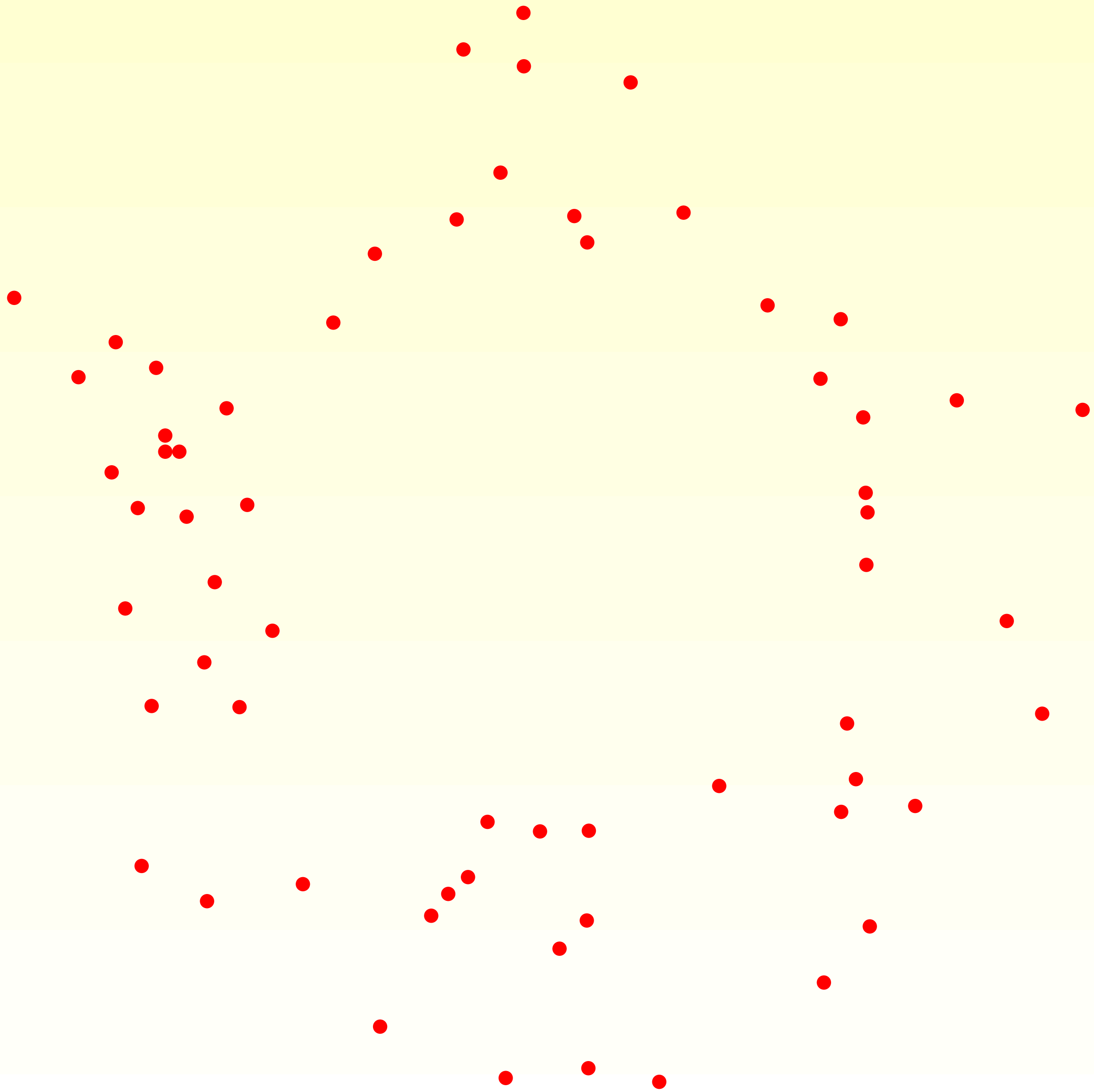

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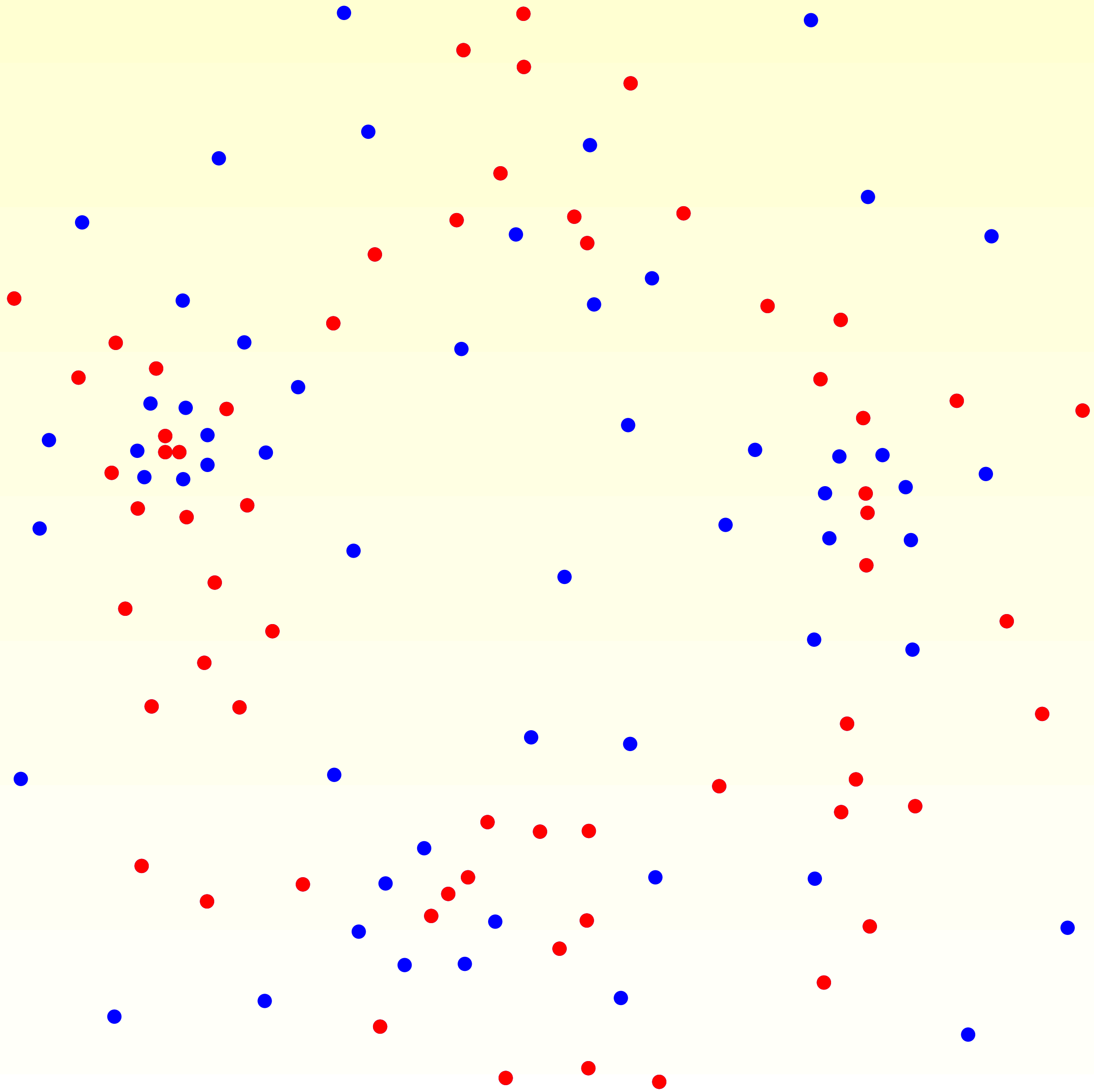
Input set of points does not have a good triangulation

We need to insert Steiner points



Mesh Refinement

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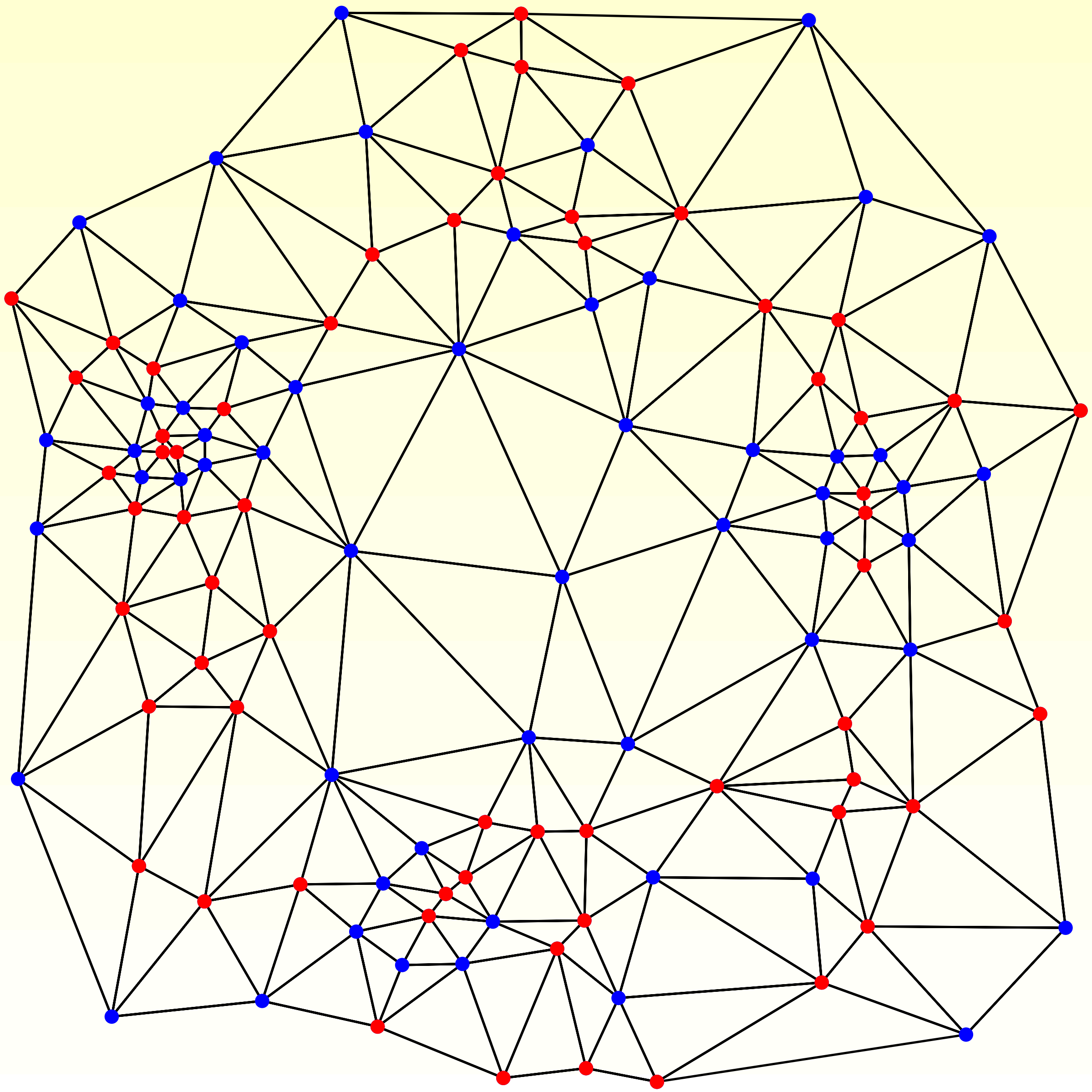
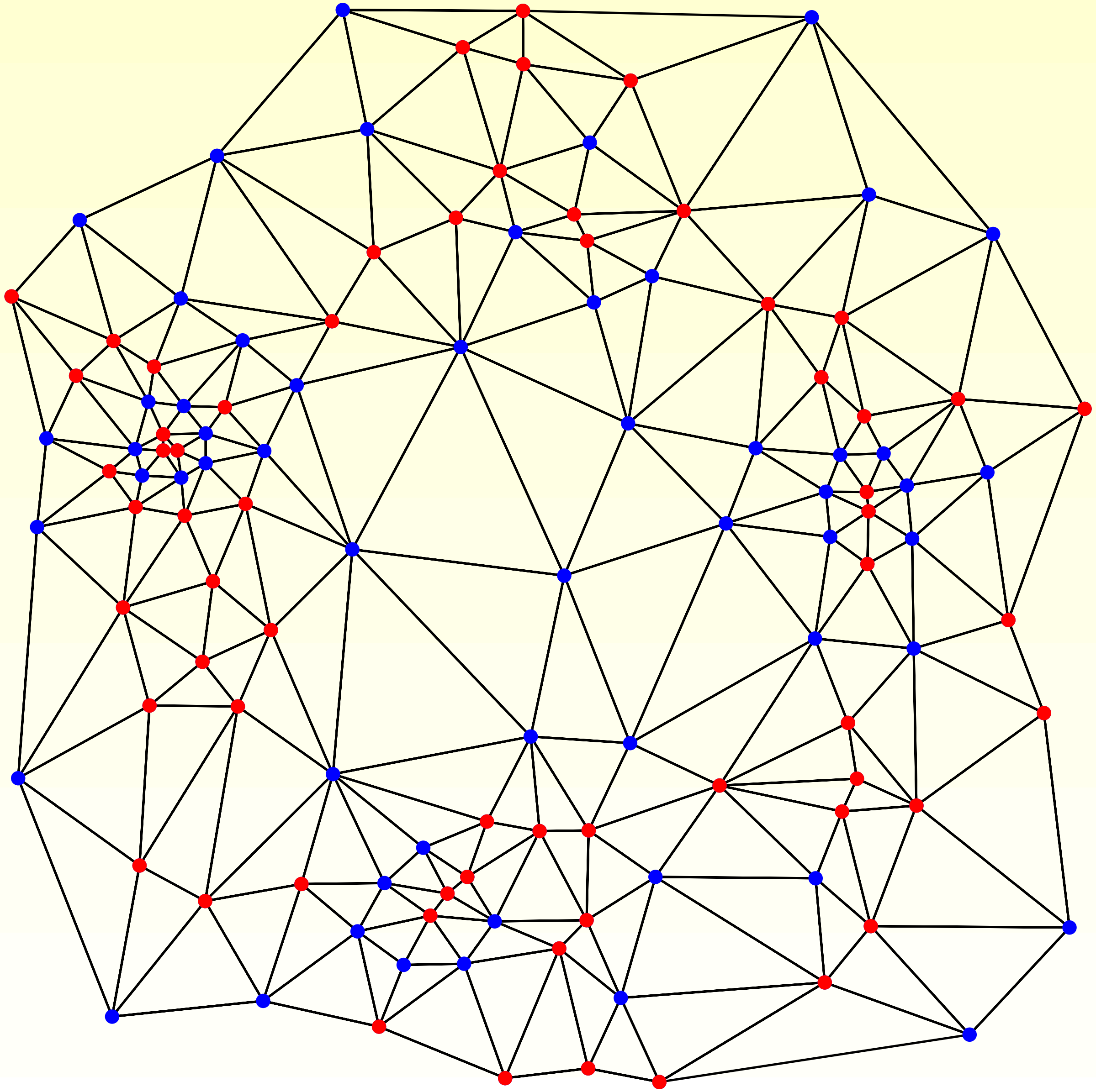
The output has a quality Delaunay triangulation, i.e., no small angles



Dynamic Mesh Refinement

Assume somebody wants to change the input set



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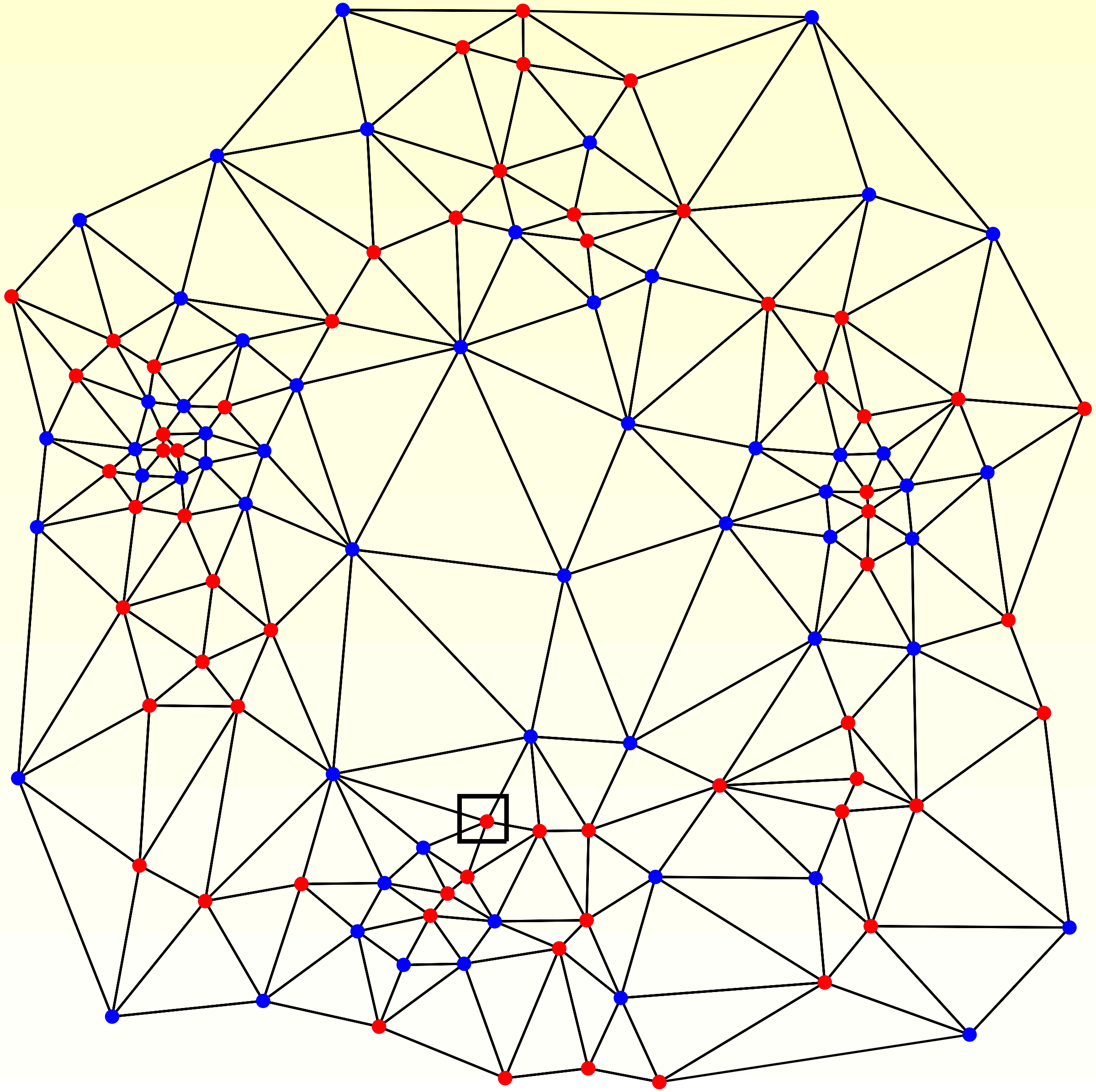
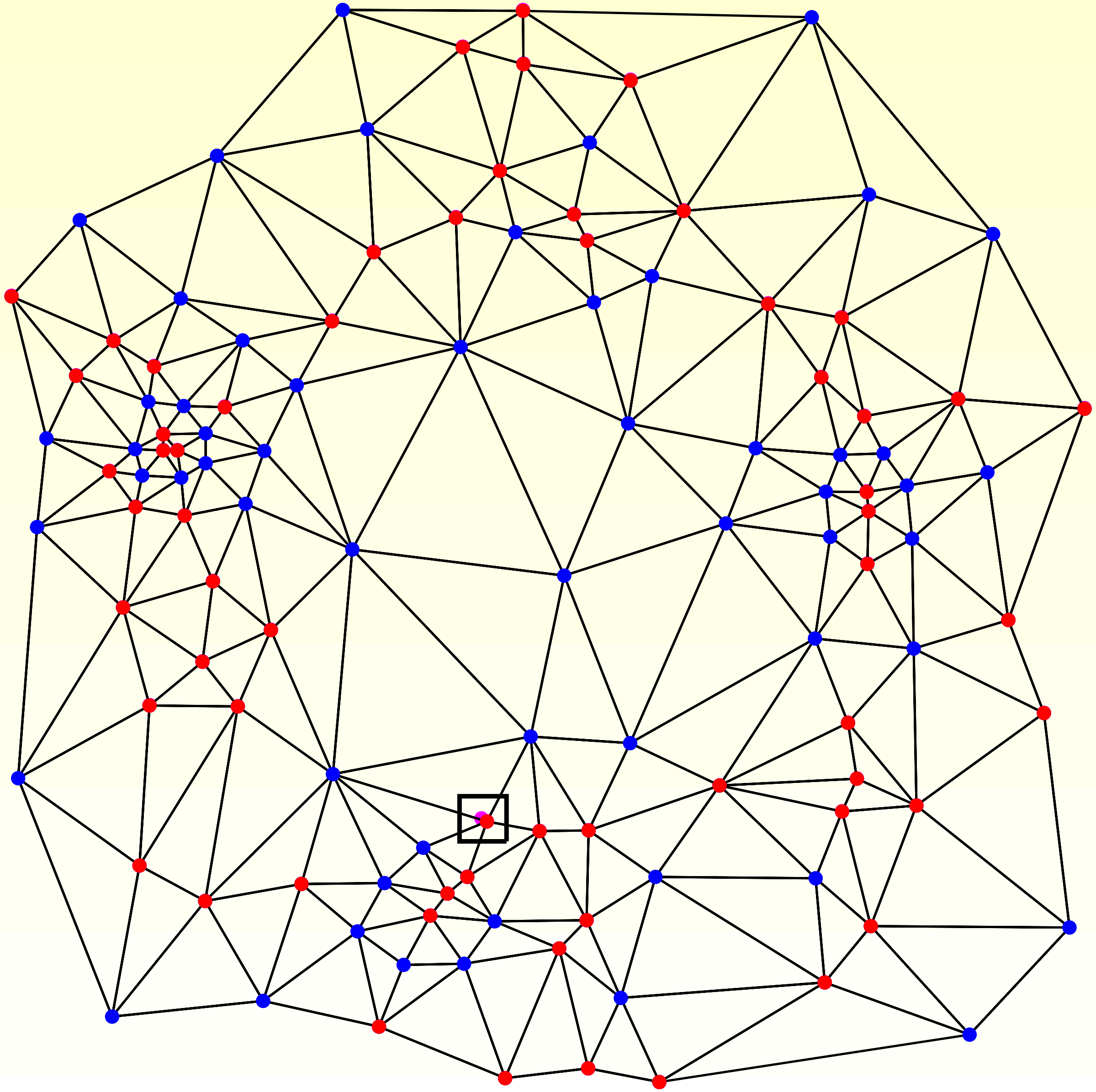


Dynamic Mesh Refinement

Assume somebody wants to change the input set

Update the set of Steiner points efficiently, without too many changes



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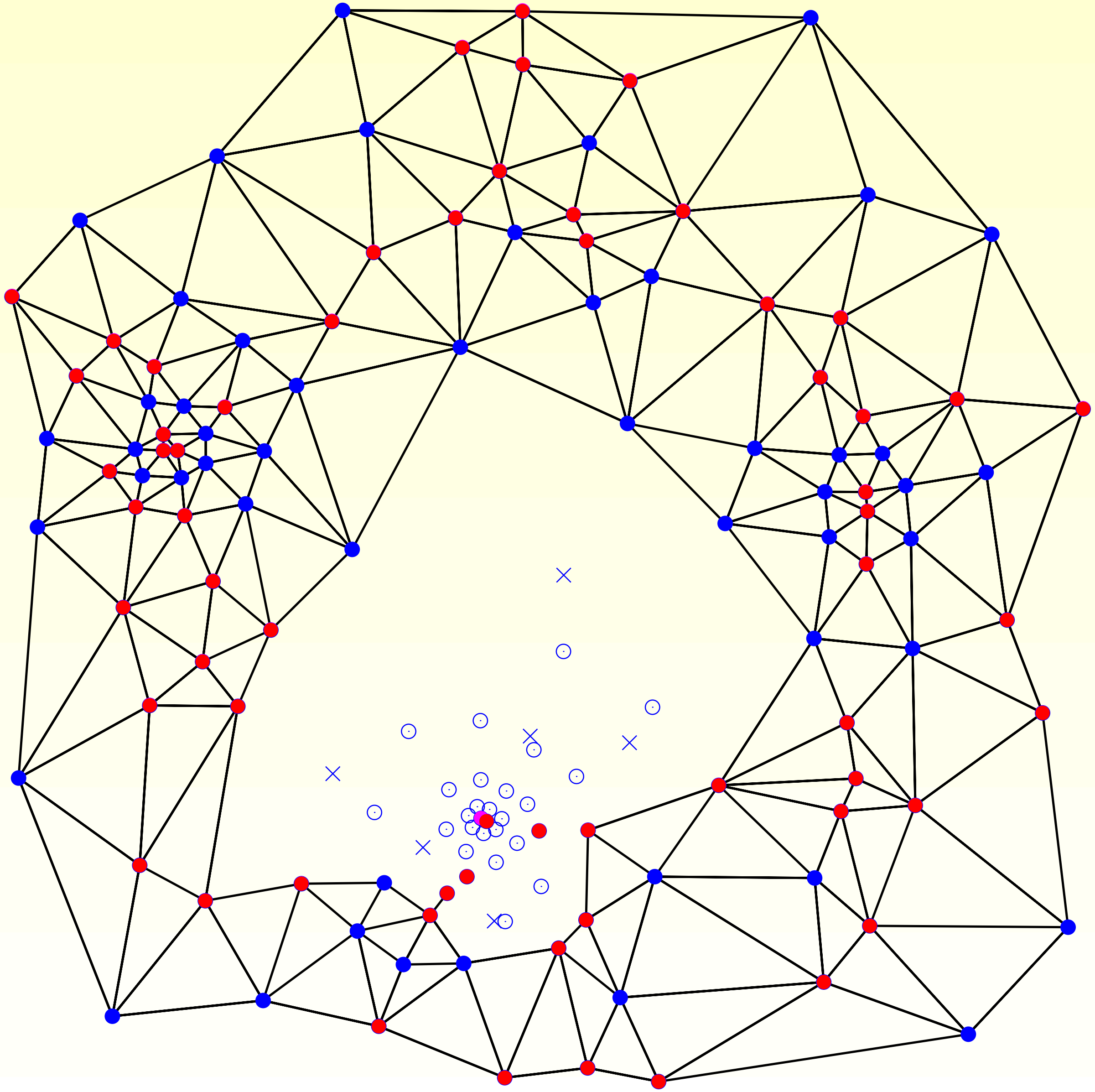
Update the set of Steiner points efficiently, without too many changes

Sustain output size to be small



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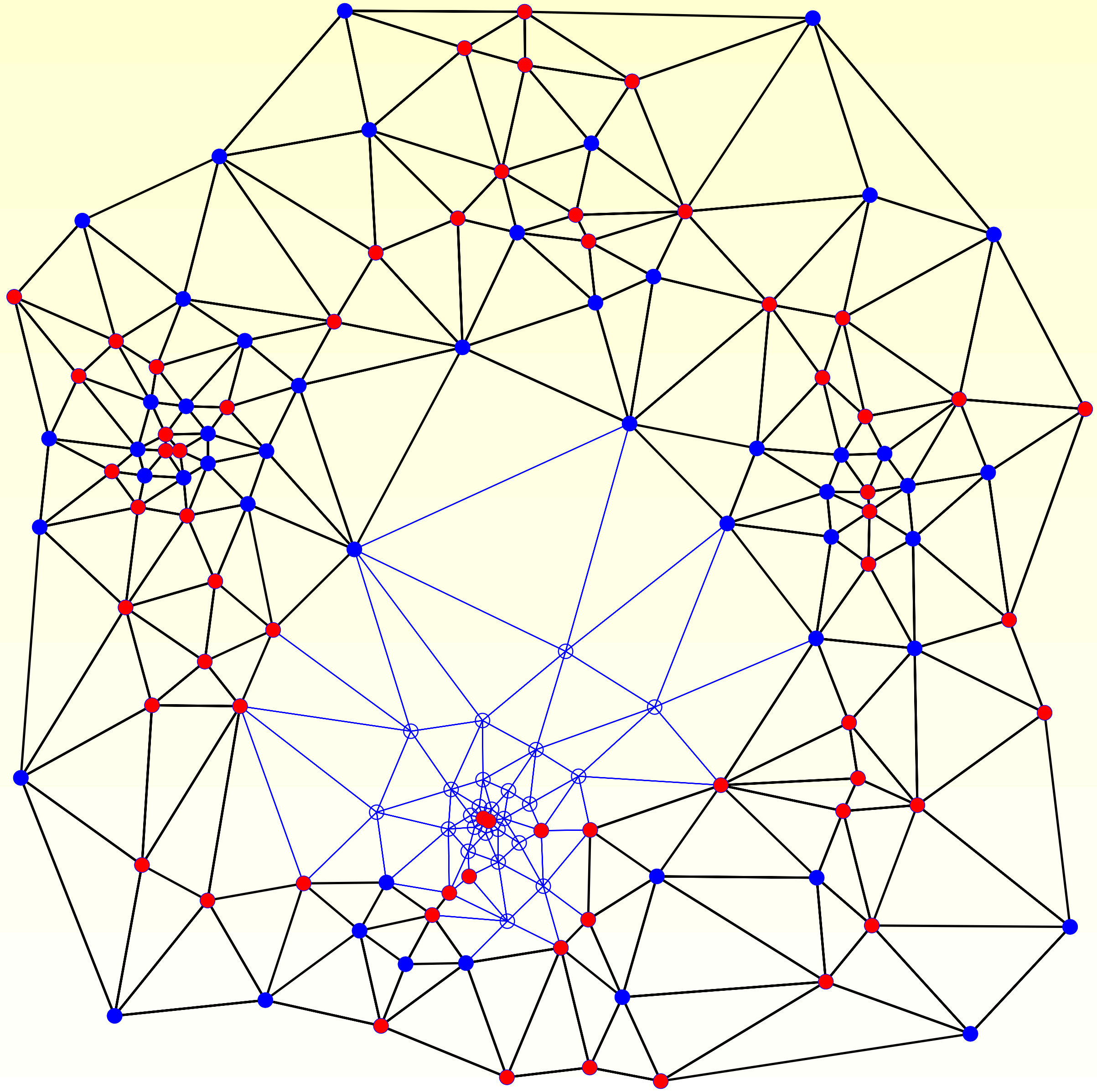
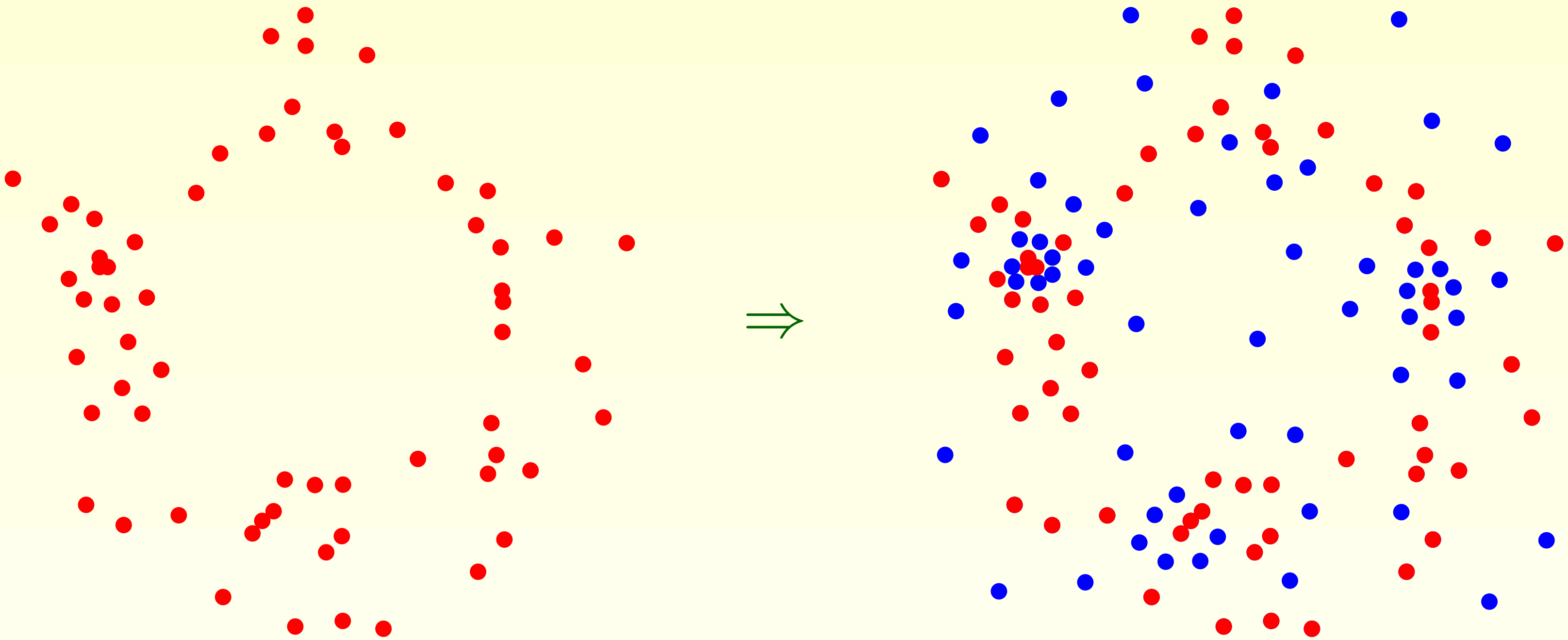


Well-Spaced Point Set Problem

Compute a ρ-well-spaced superset of a given input



Maintain a ρ-well-spaced superset as input changes

Objectives: Efficiency and Size-Optimality

Related Work

- Chew '89 (First non-heuristic construction algorithm)
- Ruppert '95 (Size-optimal algorithm in 2D)
- Spielman, Teng, Üngör '02 (Parallel algorithm in 2D & 3D)
- Har-Peled, Üngör '05 (Efficient algorithm in 2D)
- Hudson, Miller, Phillips '06 (Efficient in arbitrary D)
- Hudson, T '08 (Precursor of this work, arbitrary D)

Existing Approaches for the Dynamic Problem

1) Update is efficient but updated output is not size-optimal

2) Worst case update time is as bad as running from scratch

Our Results

Construction Algorithm

- Given N and $\rho > 1$, output $M \supset N$ is ρ-well-spaced
- Output is **size-optimal** w.r.t. N, i.e. $|M| = O(|M_{OPT}|)$
- Runs in $O(n \log \Delta)$ time, $\Delta = \frac{\text{diameter}}{\text{smallest distance}}$

Dynamic Update Algorithm

- Given an insertion/deletion $(N \to N')$, modifies the output
- Modified output M' is **size-optimal** w.r.t. N'
- Update in $O(\log \Delta)$ time $\Rightarrow |M' \ominus M| = O(\log \Delta)$
- Worst case lower bound requires $|M' \ominus M| = \Omega(\log \Delta)$

Our Solution — Change Propagation

Stable Algorithm: Executions with similar inputs should produce similar outputs/intermediate data

$$
\begin{array}{ccc}
{\color{green}N} & \xrightarrow{\varepsilon\text{-modification}} & {\color{green}N'} \\
{\color{red}\Downarrow} & \longrightarrow & {\color{red}\Downarrow} \\
{\color{red}M} & \xrightarrow{\varepsilon\text{-change}} & {\color{red}M'}
\end{array}
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Offline Algorithm $\Rightarrow$ Quality

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$$\textbf{Offline Algorithm} \Rightarrow \textbf{Quality}$$

$$+$$

$$\textbf{Stability} \Rightarrow \begin{cases} \textbf{Executions are similar} \\ M' \ominus M \textbf{ is small} \end{cases}$$

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Offline Algorithm $\Rightarrow$ **Quality**

$+$

$$\textbf{Stability} \Rightarrow \begin{cases} \textbf{Executions are similar} \\ M' \ominus M \textbf{ is small} \end{cases}$$

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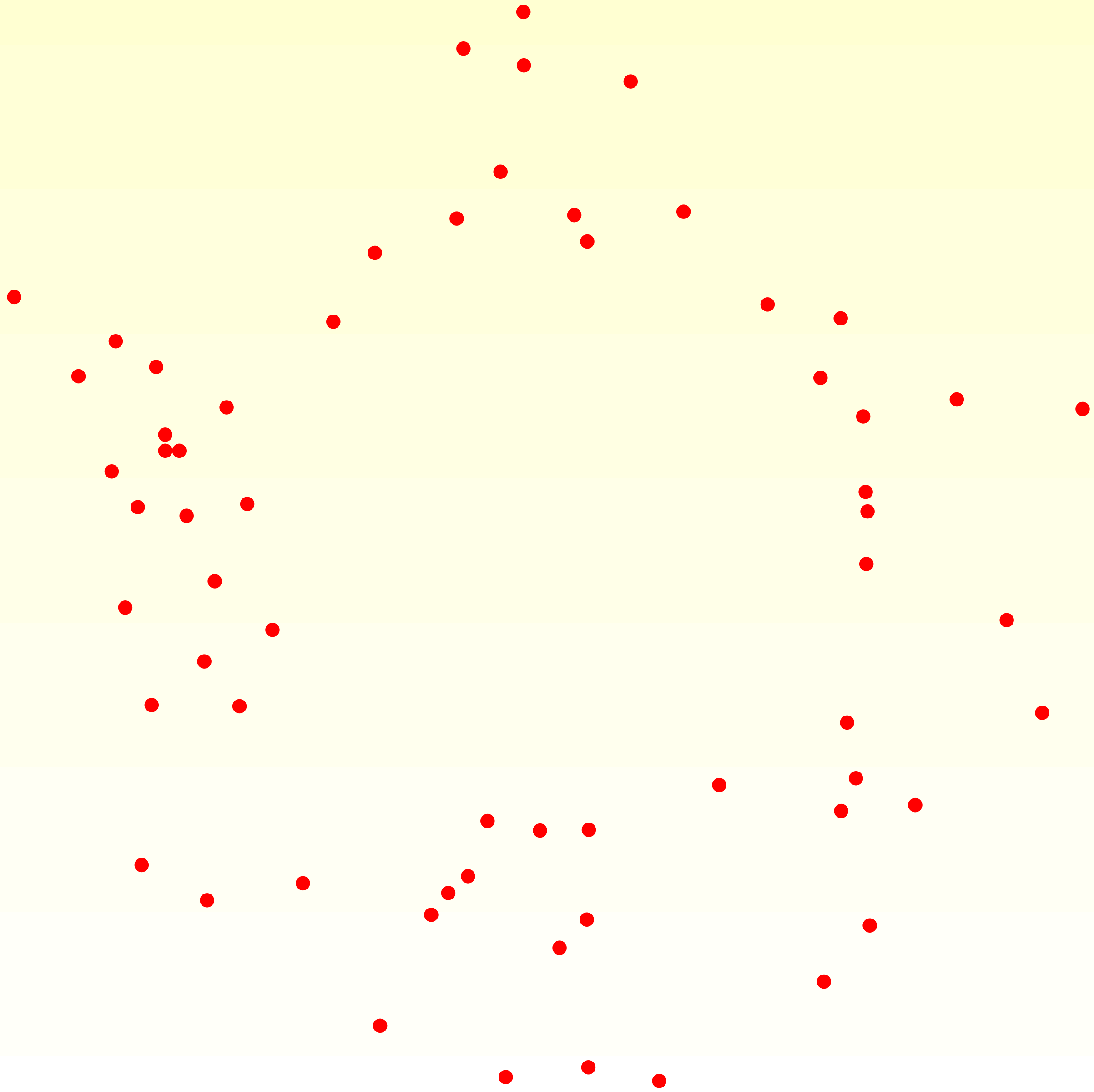
Dynamizing Stable Algorithm $\Rightarrow$ **Efficiency**

Stable Refinement Algorithm

Basic operation of the construction algorithm is fill



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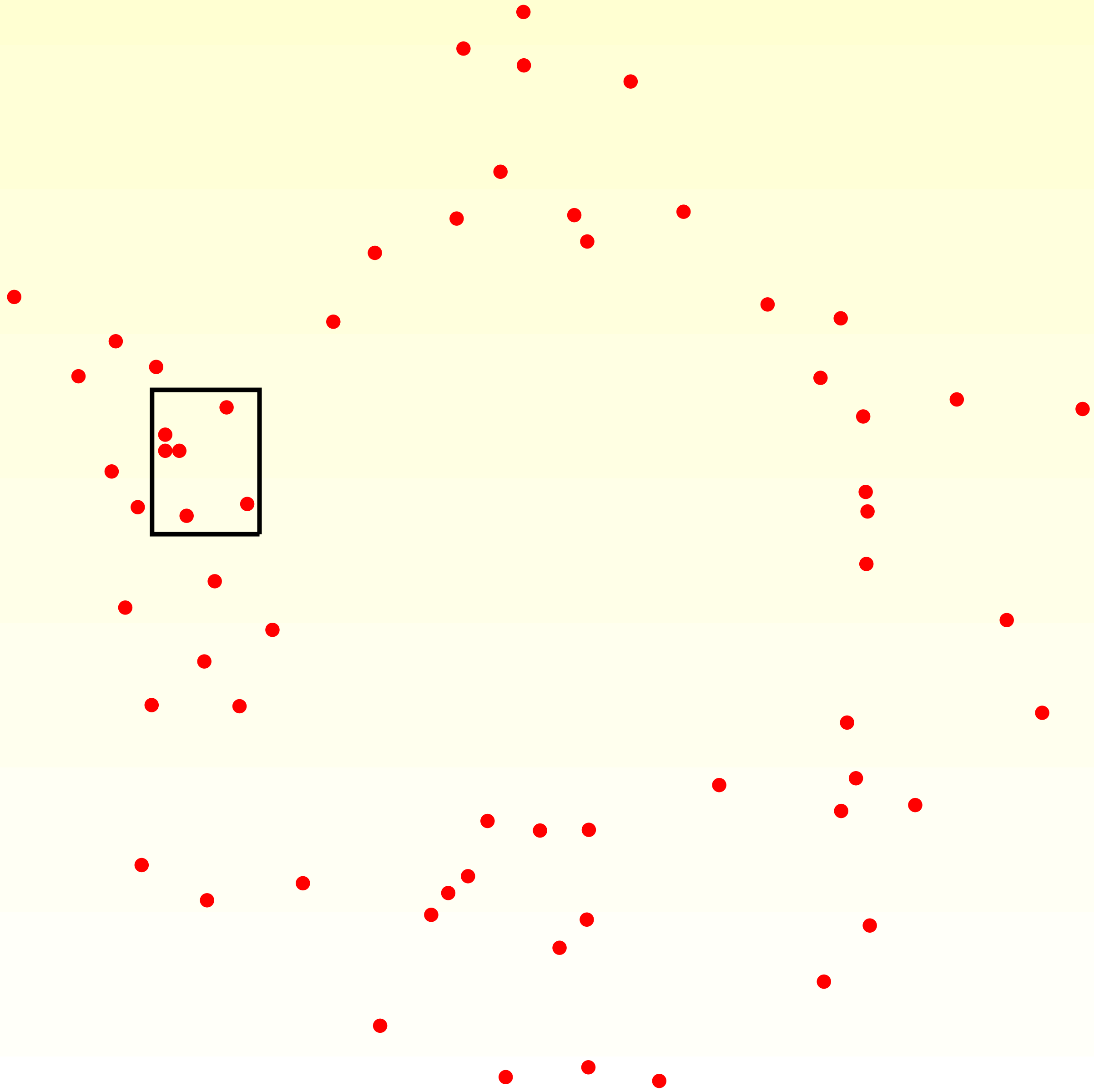

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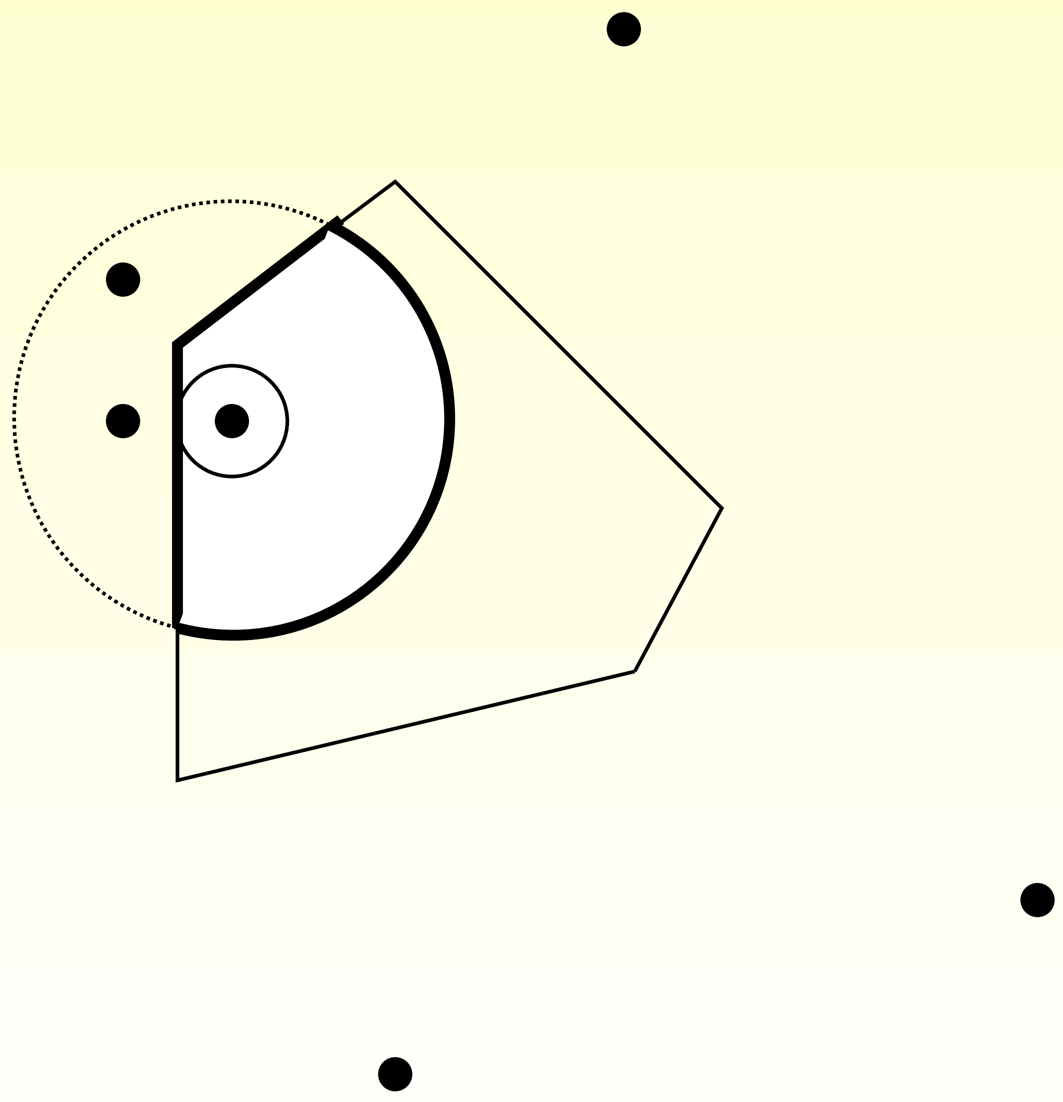

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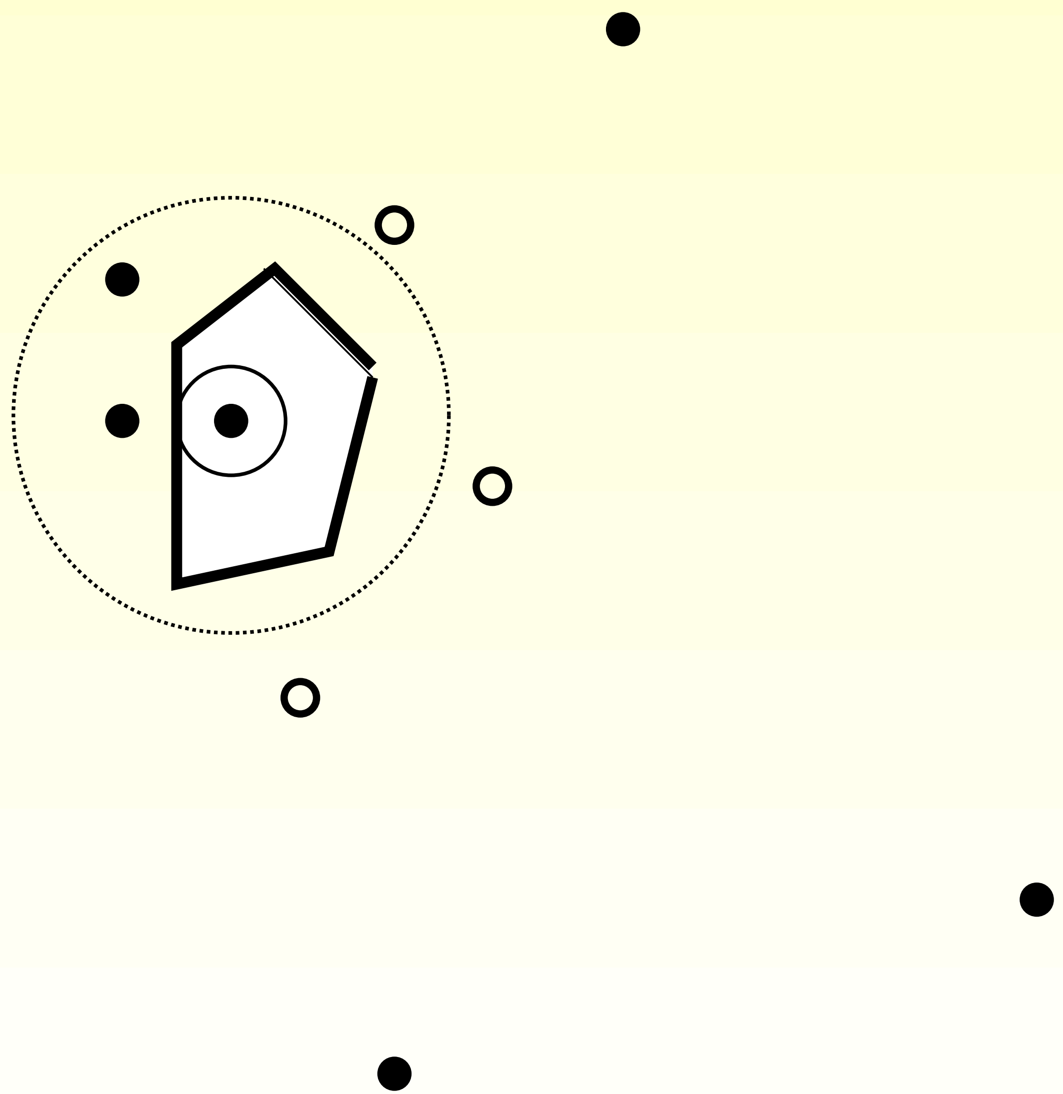

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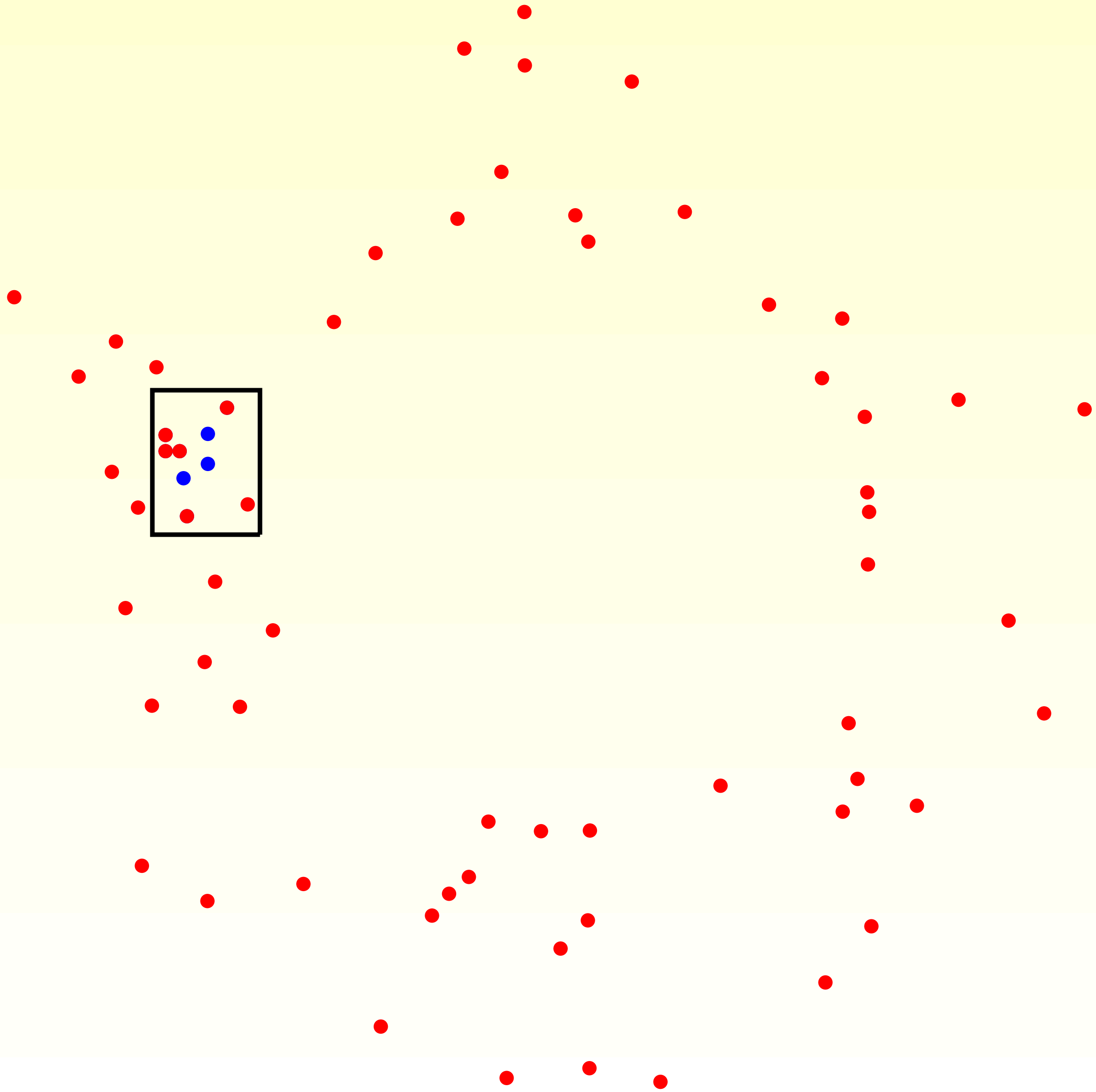

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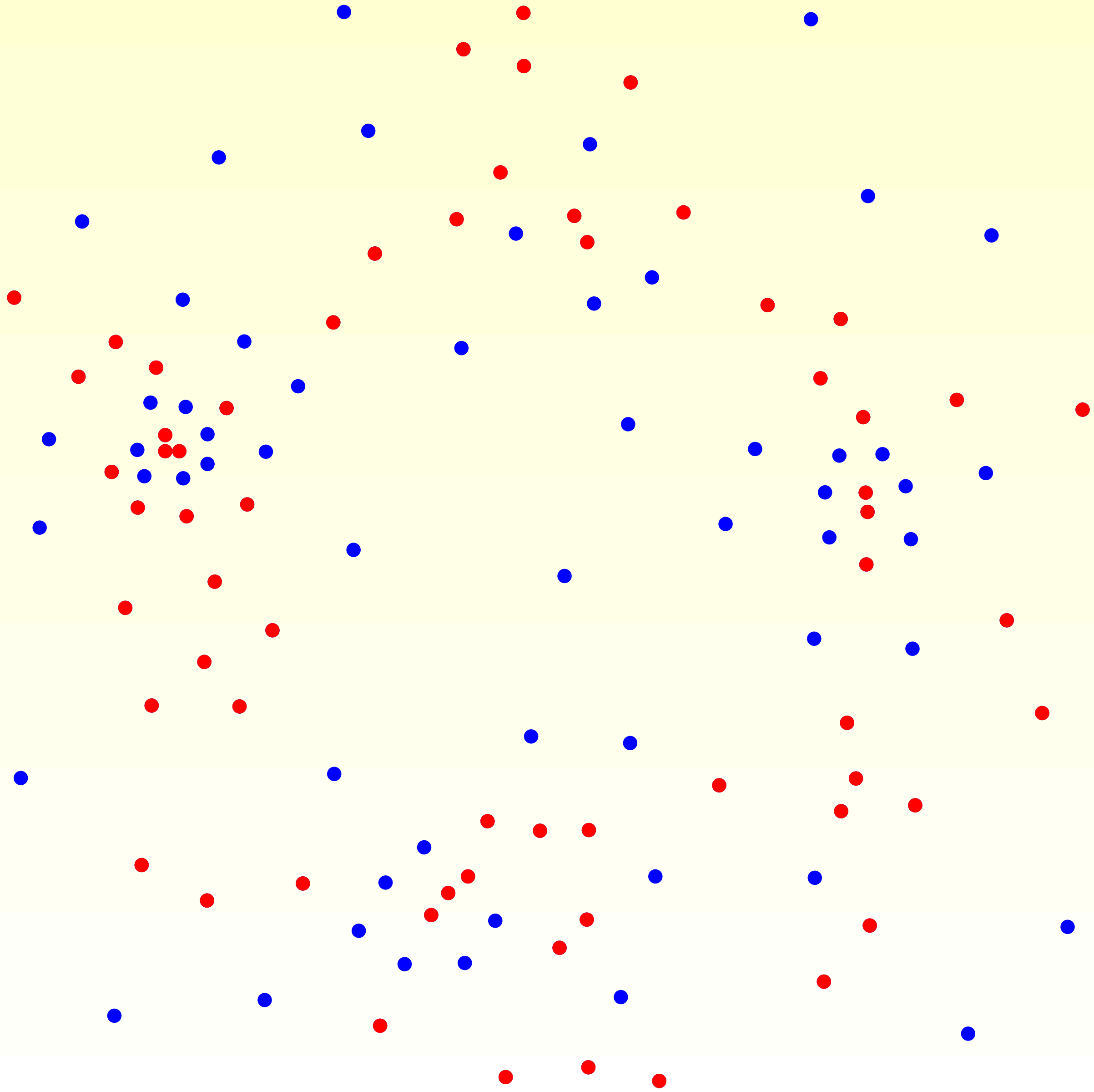


Fill Operation and Picking Region

Goal: Ensure that filling points in a single pass is sufficient

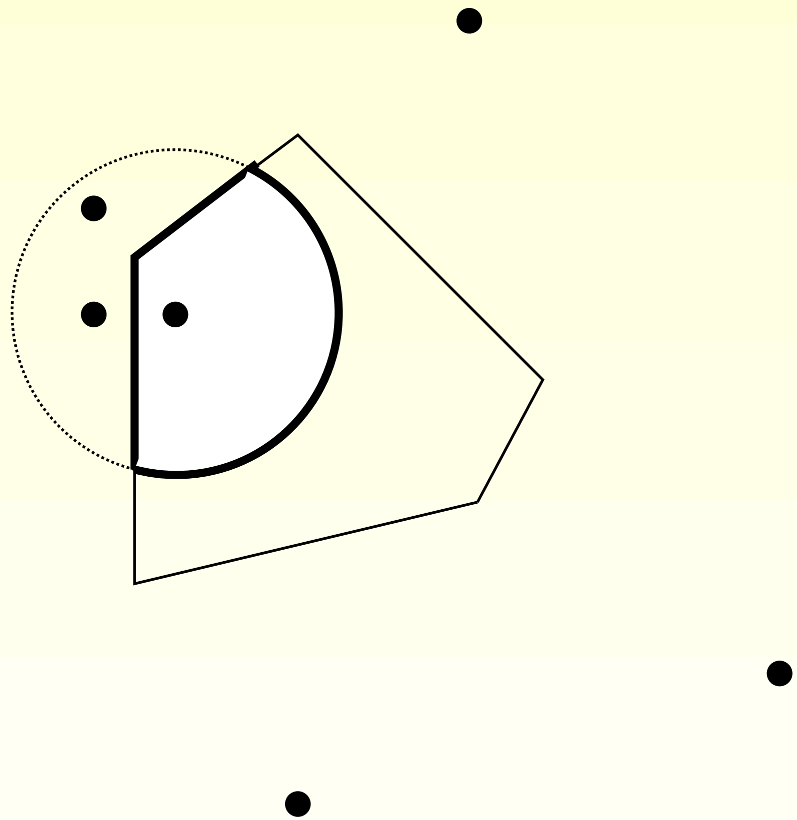
Approach: Expansion

Insert Steiner points that have ρ times larger empty balls

Goal: Efficiency, Independence

Approach: Clipped Voronoi

Location of Steiner points depend only on close Voronoi neighbors



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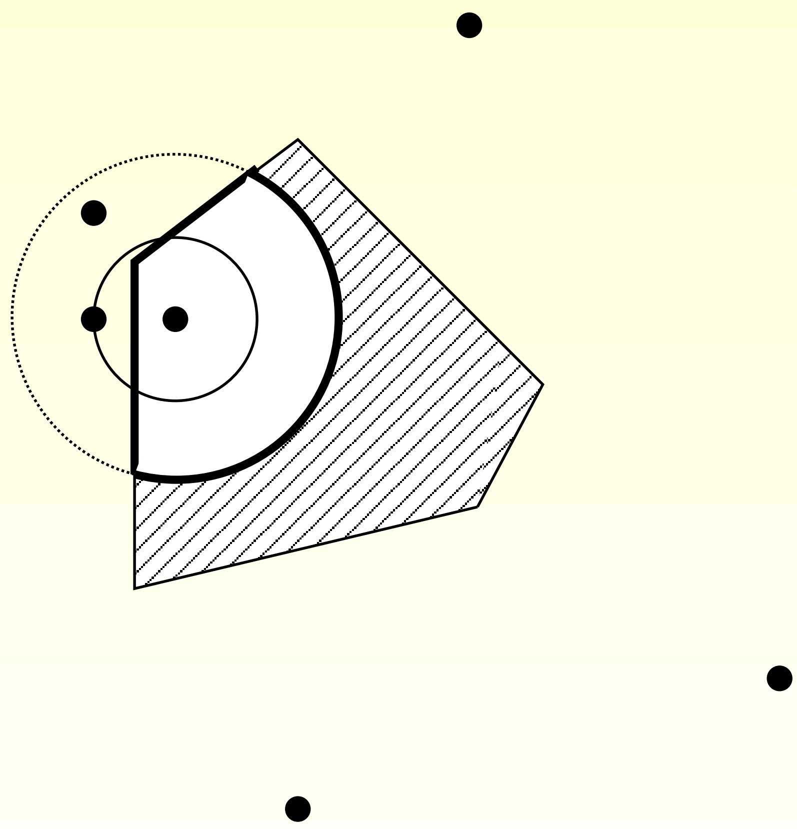

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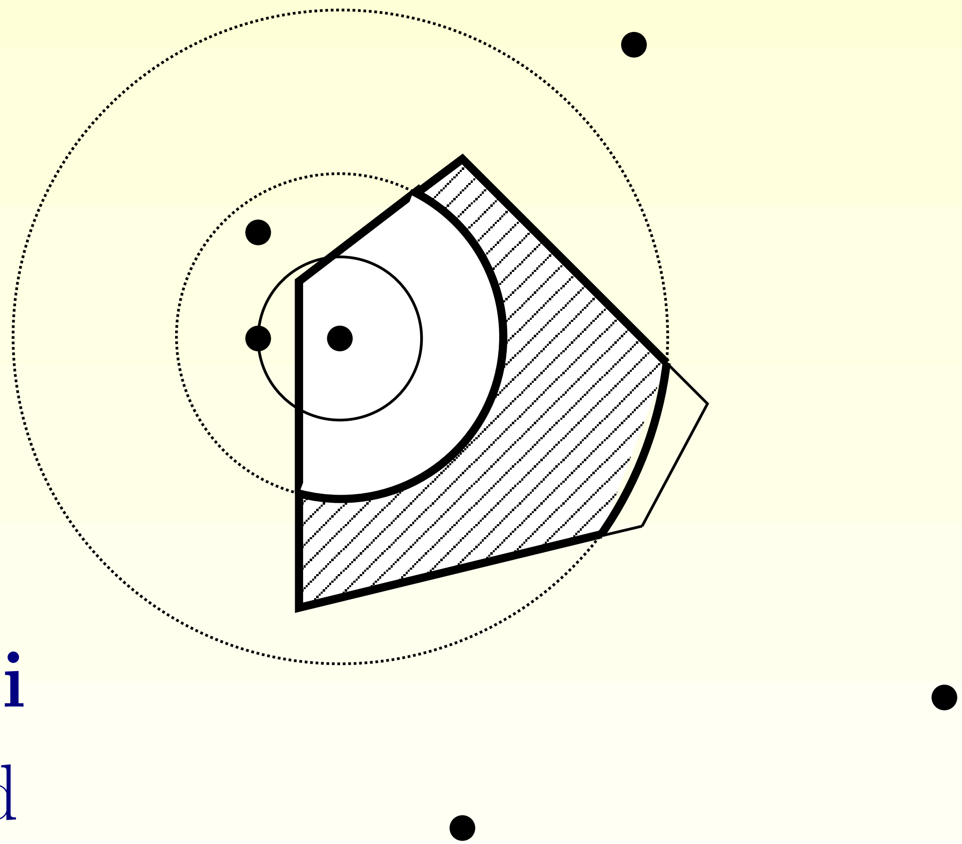

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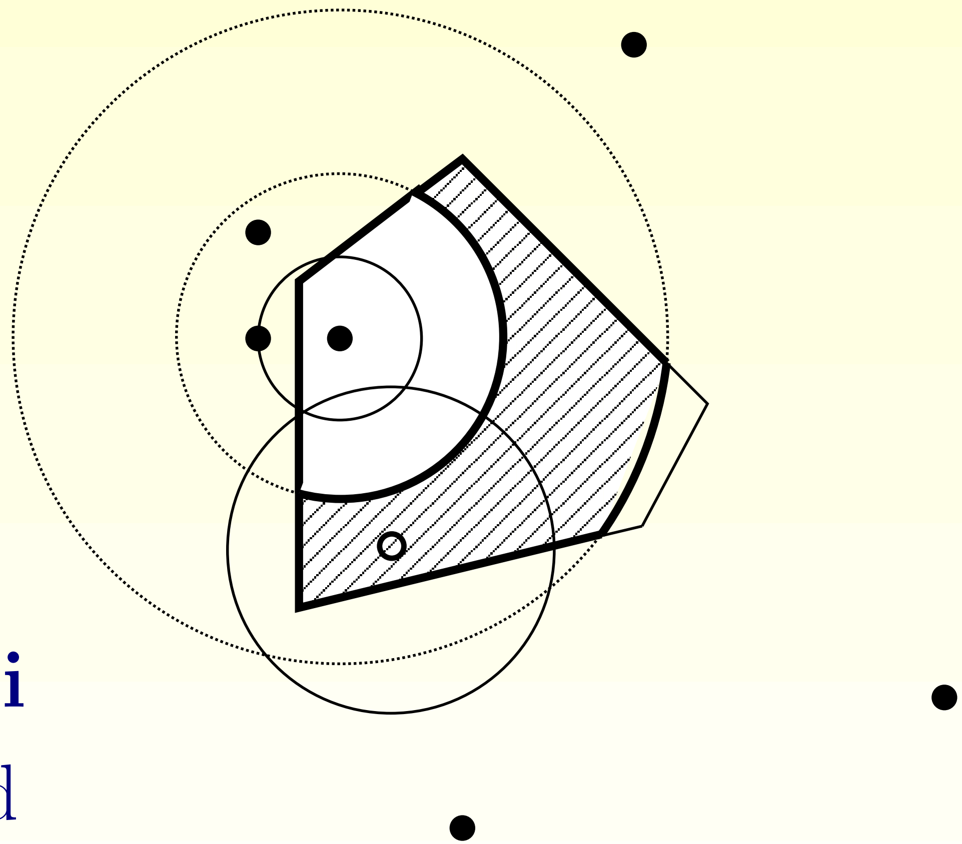

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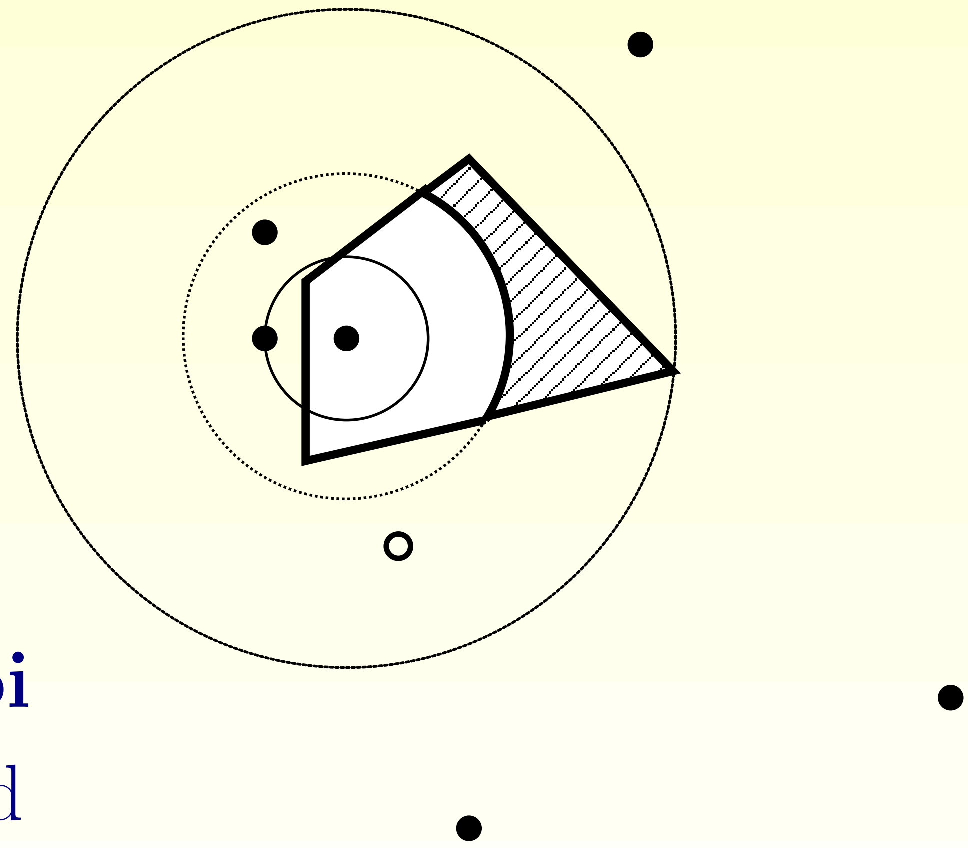

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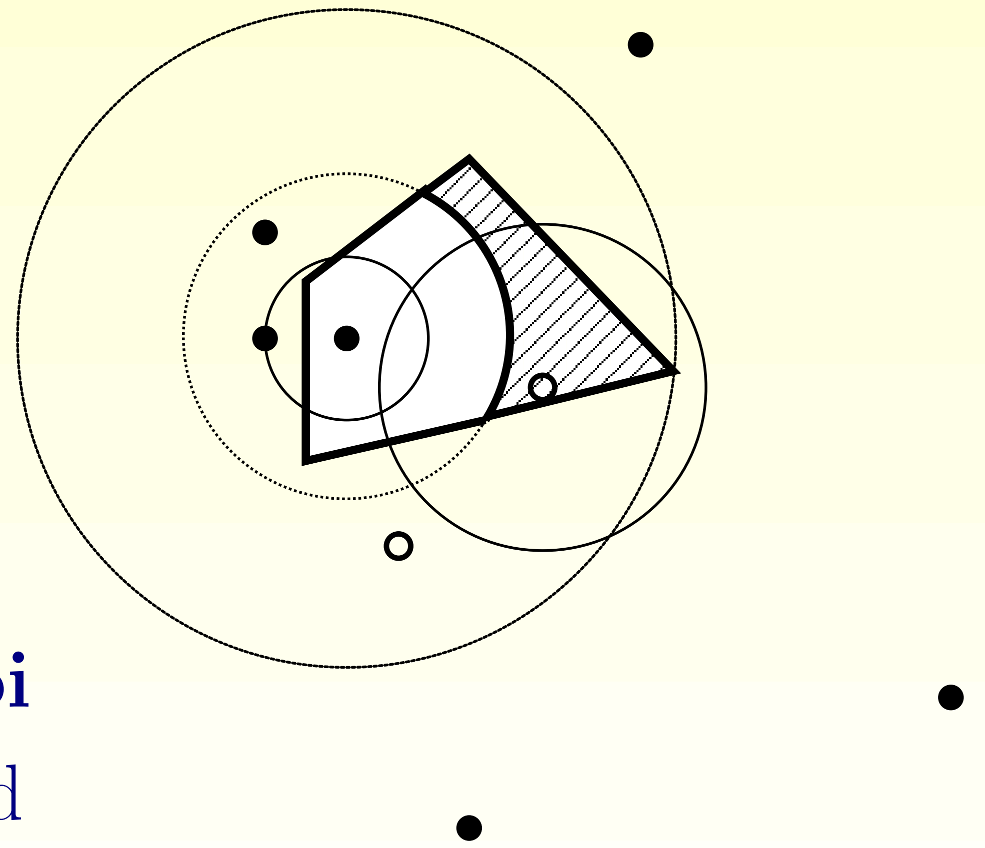

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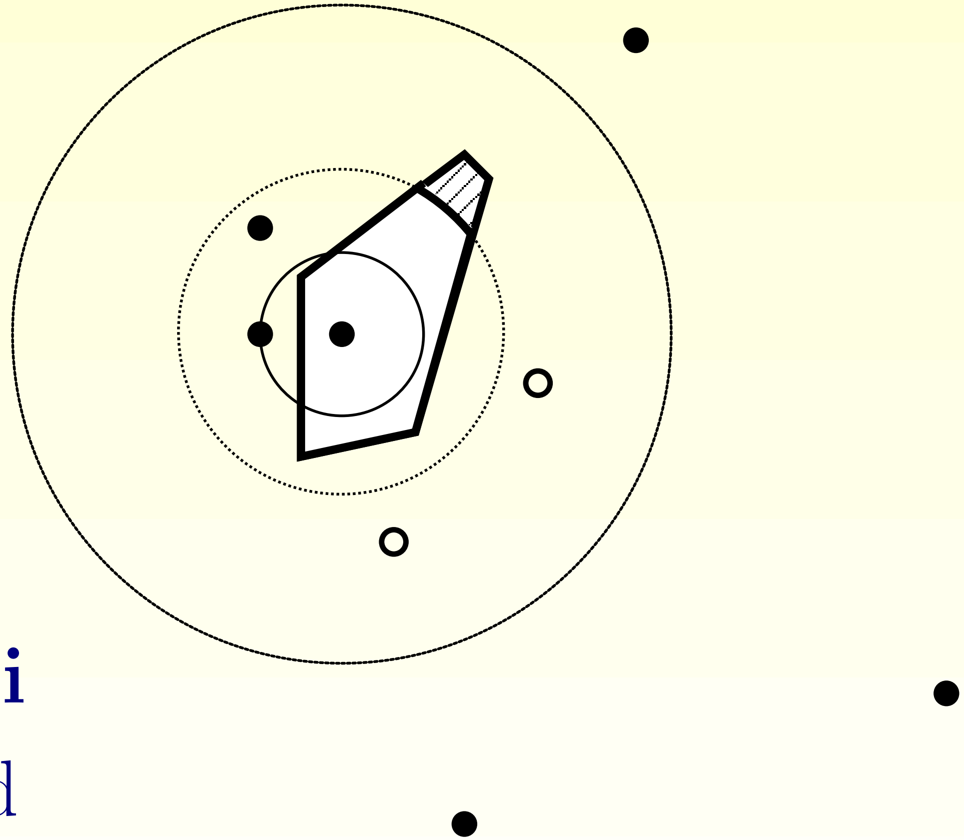

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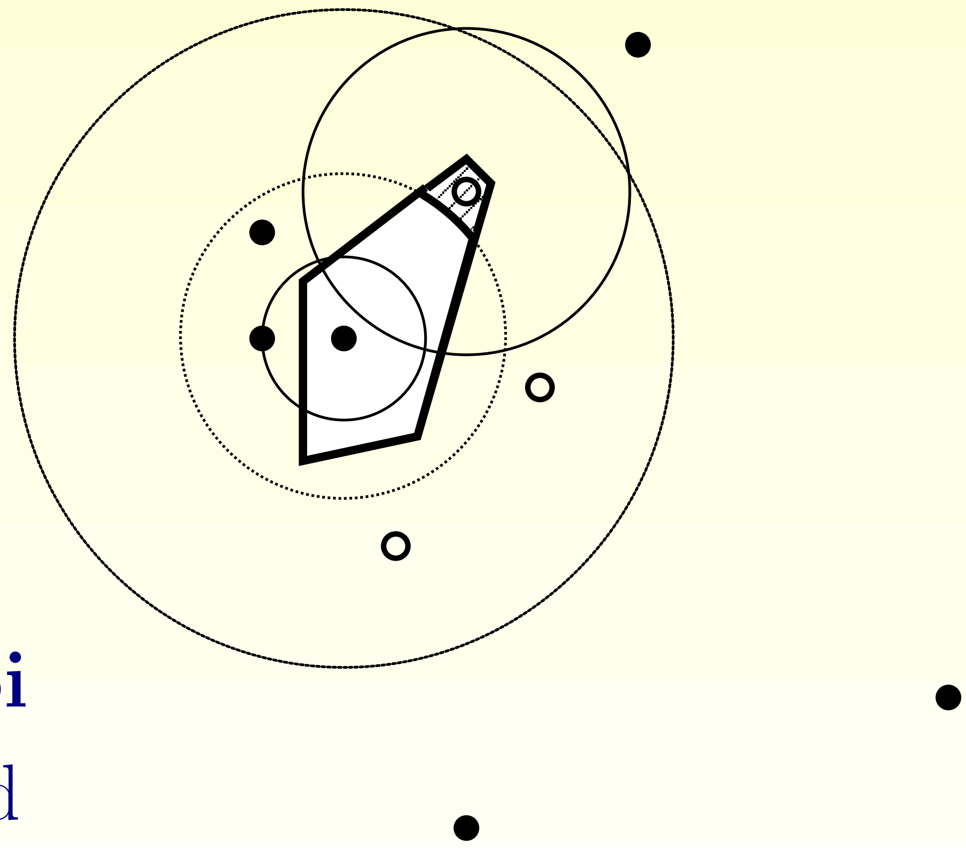

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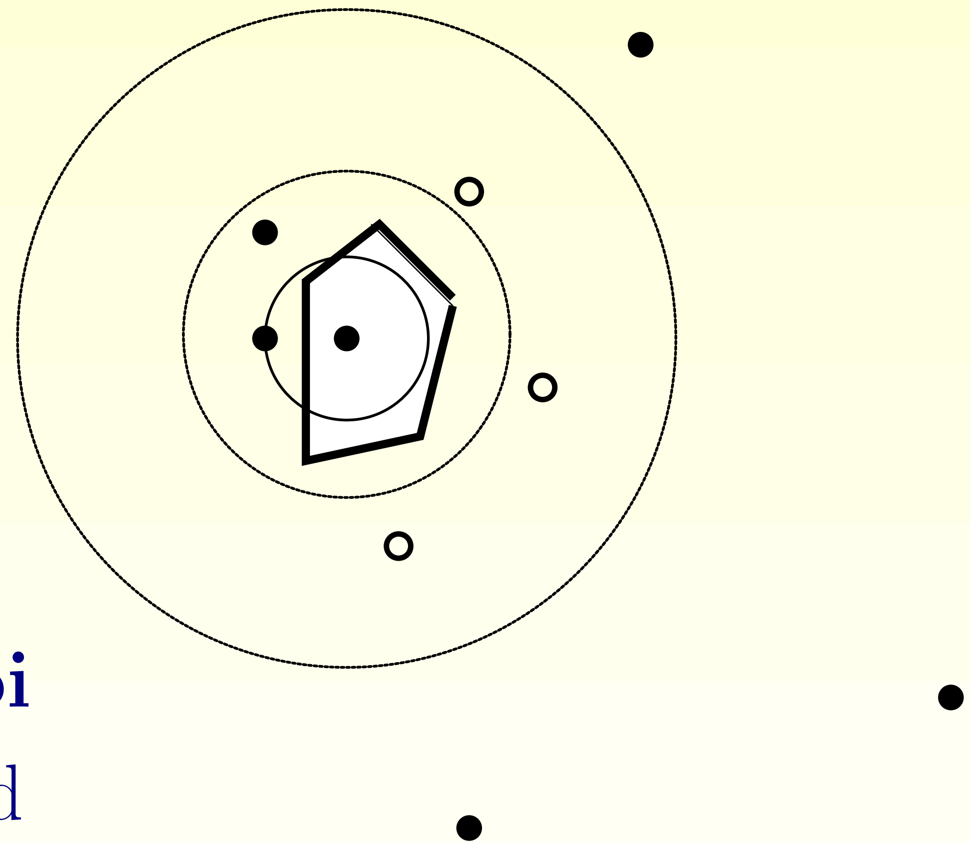
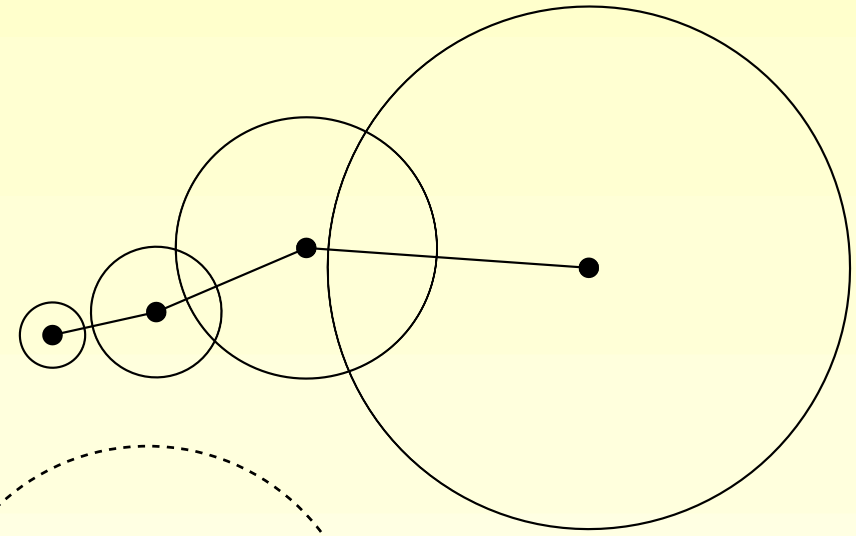


Key Ideas for Stability

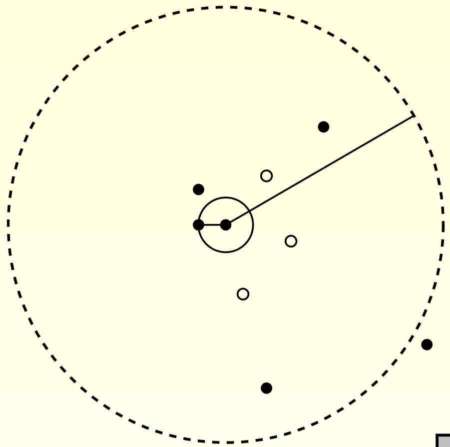
Expansion

Steiner points have ρ times larger empty balls



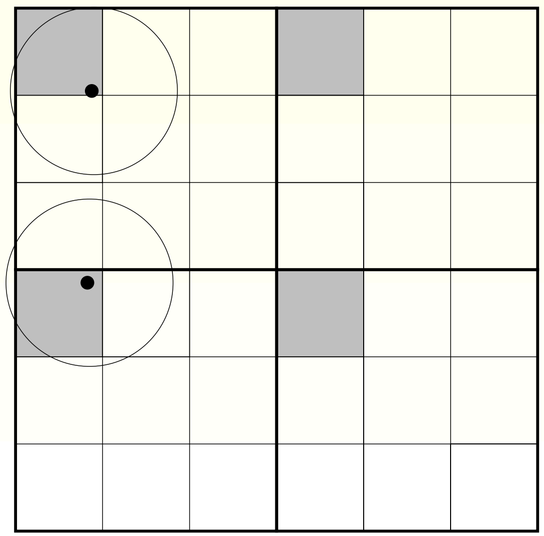
Locality

Steiner point selection depends on a bounded neighborhood



Independence

At each rank, filling distant points are guaranteed not to affect each other



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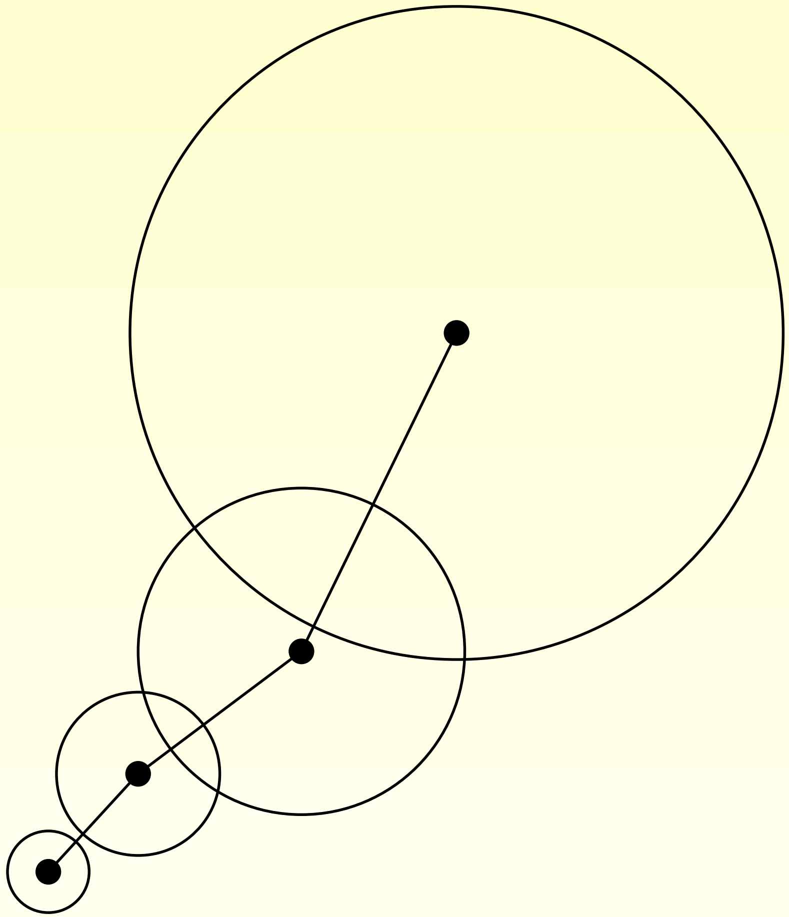

Rank = logarithm base ρ of nearest neighbor distance

In total $O(\log \Delta)$ ranks

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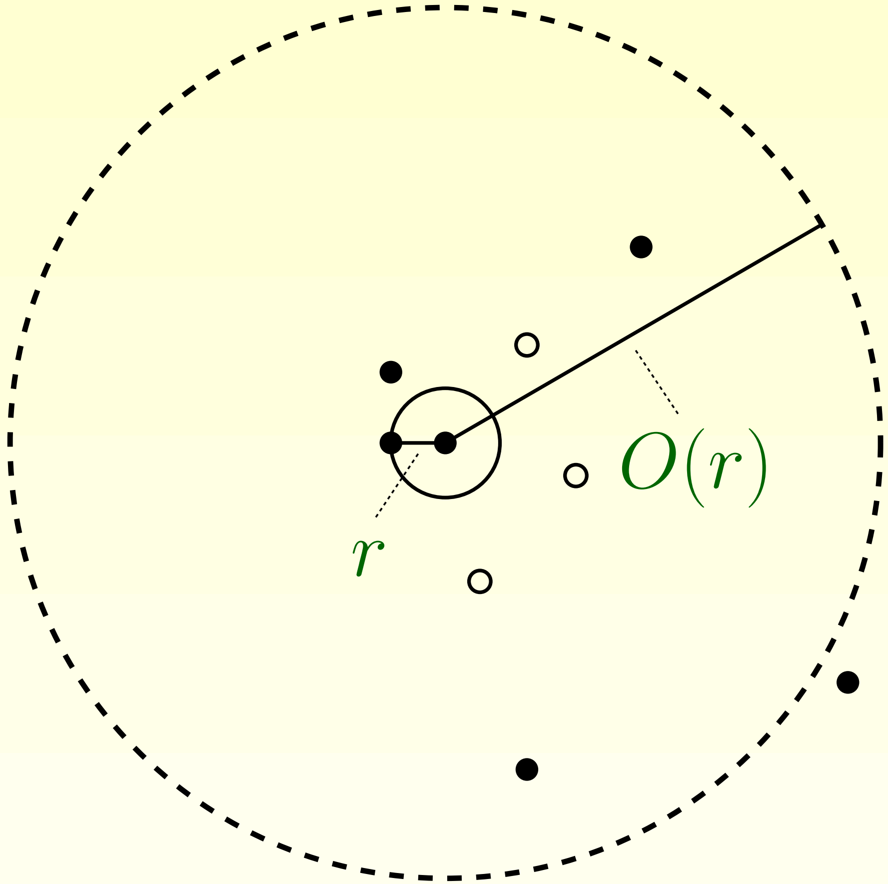

Points outside the big ball do not influence how we pick Steiner points

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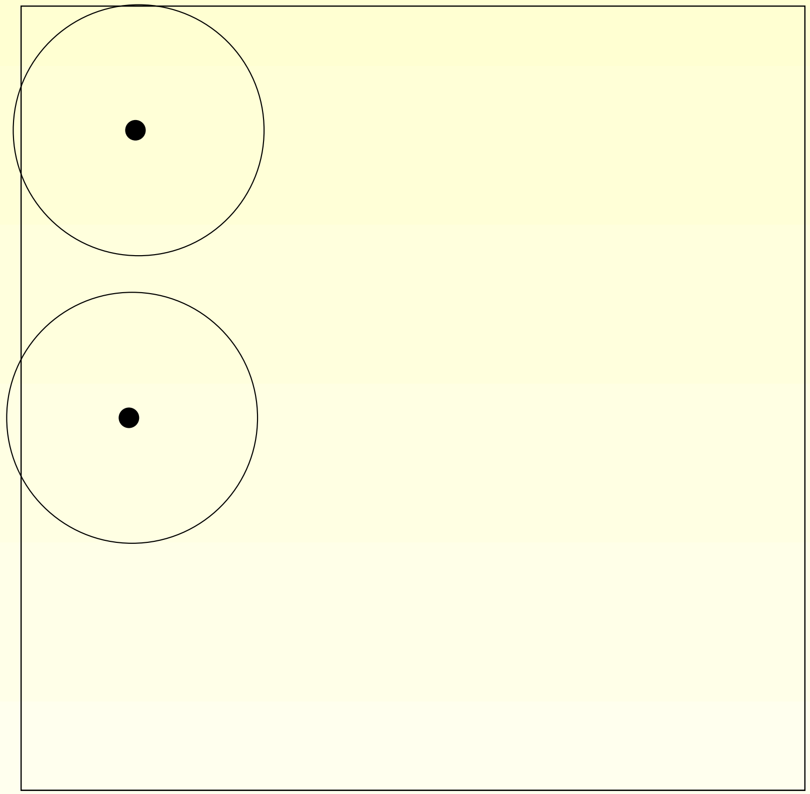

Process points in rank order

Two points at a given rank may not depend on each other

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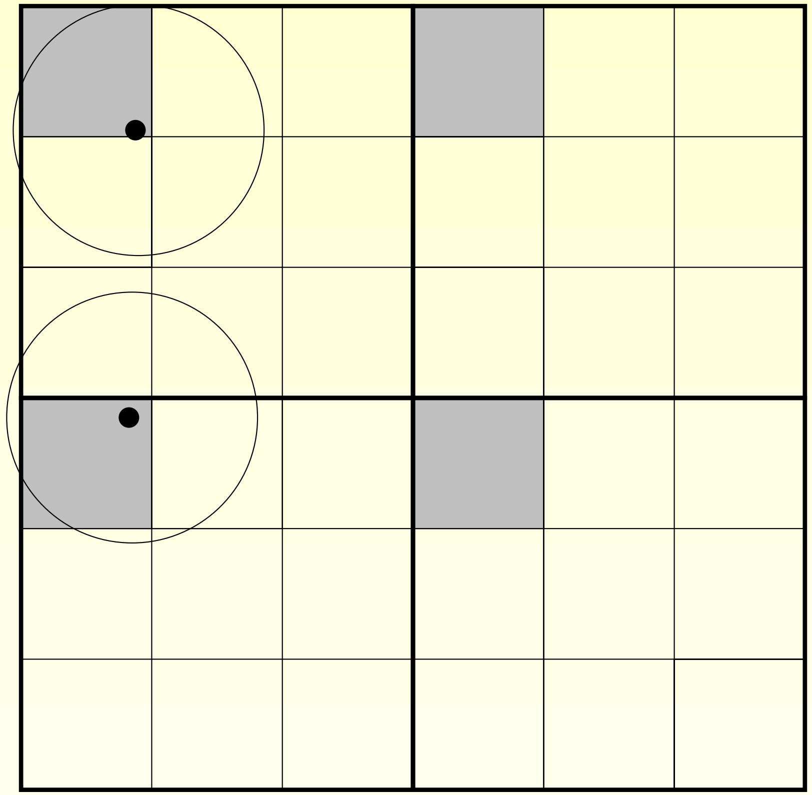


Independence

At each rank, filling distant points are guaranteed not to affect each other

For ensuring independence partition space using $O(1)$ colors and process points in color order

Proof of Stability — Dependency Path

Schedule using Ranks and Colors

Algorithm processes points in stages by ordering them first by rank then by color

Dependency Paths

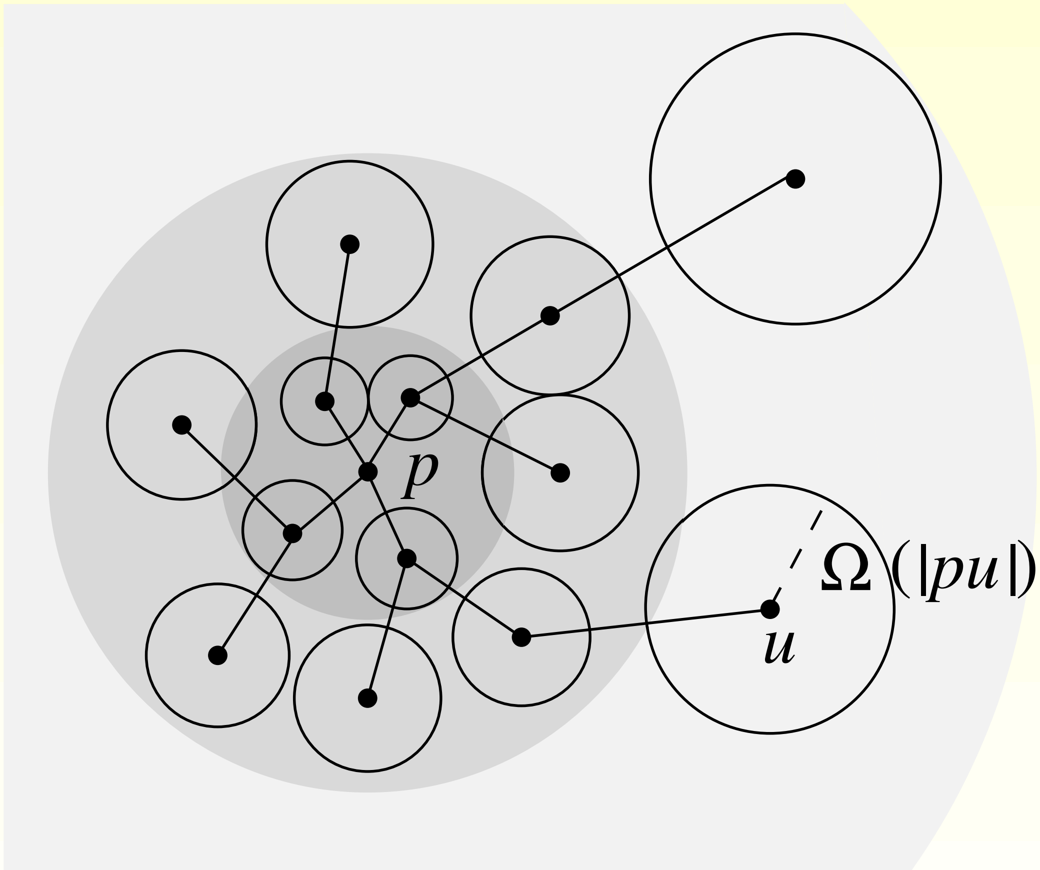
Expansion and independence guarantee that filling a point does not affect points processed in earlier stages and the points being processed at the current stage

There are $O(\log \Delta)$ stages in total

Proof of Stability — Spacing and Packing

Assume inserting (or deleting) a point p

Path: Given a point u at rank r if there is a dependency path from p to u, then $|pu| < O(\rho^r)$



Spacing: There is an empty ball around u of radius $\Omega(\rho^r)$ which consequently is $\Omega(|pu|)$

Packing: There can be only $O(1)$ such points at each rank $O(\log \Delta)$ in total

Thank You!

Questions?