PS3 Solutions

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Problem 1

By definition:

$$\begin{aligned}
(\nabla g)(x)(y) &= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(g(x+\epsilon y)-g(x)\right)\\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left((x+\epsilon y)M(x+\epsilon y)-xMx\right)\\
&= \lim_{\epsilon\to 0}\left(yMx+xMy+\epsilon yMy\right)\\
&= yMx+xMy
\end{aligned}$$

So in order to write an expression for ∇g, we must "pull out" the matrix M from the function g. This may be accomplished by finding a bilinear form which agrees with g, as follows:

$$g' = \text{the}\left(f : V\to V\to\mathbb{R}\,\text{s.t.}\,\forall x : V.\left(f(x)(x)=g(x)\right)\wedge\text{IsBilinear}(f)\right)$$

Inside the "the" expression, forcing f to be bilinear causes it to have the form $f(x)(y)=xM'y$ for some matrix M', while the first condition forces $xM'x=xMx$ for all x, which implies that $M'=M$ (because if $x(M'-M)x=0$ for all x, then $M'-M=0$). Hence, we need only define IsBilinear:

$$\begin{aligned}
\text{IsBilinear} &= \lambda f : V\to V\to\mathbb{R}.\forall x : V.\forall y : V.\forall z : V.\forall\alpha : \mathbb{R}.\\
&\quad\left(\left(f(\alpha x+z)(y)=\alpha f(x)(y)+f(z)(y)\right)\wedge\left(f(x)(\alpha y+z)=\alpha f(x)(y)+f(x)(z)\right)\right)
\end{aligned}$$

And we may now define ∇g:

$$\nabla g = \lambda x : V.\lambda y : V.\left(g'(x)+g'(y)\right)$$

Problem 2

V and V^{**} are equivalent

We wish to find an equivalence $f : V\to V^{**}$ and $g : V^{**}\to V$. Since elements of V^* act on elements on V (that is, they are linear functions from V to $\mathbb{R}$), while elements of V^{**} act on elements on V^*, it would be natural to define f in such a way that for each $x\in V$, the corresponding $f(x)\in V^{**}$ acts on all elements on V^* in the same way as those elements act on x. Symbolically:

$$\begin{aligned}
f_1 &= \lambda x : V.\lambda y : V^*.y(x)\\
g_1 &= \lambda x : V^{**}.\text{the}\left(y : V\,\text{s.t.}\,f_1(y)=x\right)
\end{aligned}$$

Note that $f_1 : V\to V^*\to\mathbb{R}$, and that the function which it returns (of type $V^*\to\mathbb{R}$) is linear, showing that in fact $f_1 : V\to V^{**}$. This function is $1-1$, since for two distinct $x, x' : V$ there must exist a $y : V^*$ such that $y(x)\neq y(x')$ (for example, if we imagine a basis expansion for V, then we could take y to be the dot product of the basis expansion of $x-x'$ with the basis expansion the parameter to y). That f_1 is onto follows from the fact that if we imagine a basis expansion for V, and the induced basis expansion for V^*, we will have that $y(x)$ is simply the dot product of the basis expansions of y and x. The fact that every linear function $V^*\to\mathbb{R}$ may be written as a dot product with some vector shows that f_1 is onto. Note that this (that f is onto) only holds for finite-dimensional V (essentially because we are only able to talk about finite sums, so that while we can meaningfully work with basis expansions of x and y in infinite dimensions, the "dot product" is an infinite sum, which we cannot handle without some notion of limits, causing this argument to break down).

V and linear $V^* \to \mathbb{R}$ are equivalent

Since the set of all linear functions $V^* \to \mathbb{R}$ is precisely V^{**}, the same method as the previous part works.

Linear $V \to V$ and bilinear $V \times V^* \to \mathbb{R}$ are equivalent

The idea here is similar:

$$\begin{aligned} f_2 &= \lambda x : V \to V.\lambda (y, z) : V \times V^*.z (x (y)) \\ g_2 &= \lambda x : V \times V^* \to \mathbb{R}.\text{the} (y : V \to V \text{ s.t. } f_2 (y) = x) \end{aligned}$$

Essentially what is being shown here is that every linear map from a vector space to itself fixes a coordinate system. Choosing a basis of V allows us to represent this map as a matrix M (note that, for a single fixed basis, the map between linear functions of type $V \to V$, and matrices M, is $1-1$ and onto), and permits us to map a $y : V$ into the dual space by taking My, and then map this and a $z : V^*$ into the reals by taking the dot product (the one induced on basis expansions on V^* by the choice of basis on V) between My and z, as zMy. Note that while this reasoning uses bases, the actual value of zMy is basis independent.

Isomorphisms $V \to V^*$, bilinear inner products $V \times V \to \mathbb{R}$ and bilinear inner products $V^* \times V^* \to \mathbb{R}$ are equivalent

Note the additional restrictions on the problem statement, compared to the problem set.

First, the equivalence between $V \to V^*$ and $V \times V \to \mathbb{R}$:

$$\begin{aligned} f_3 &= \lambda x : V \to V^*.\lambda (y, z) : V \times V.x (y) (z) \\ g_3 &= \lambda x : V \times V \to \mathbb{R}.\text{the} (y : V \to V^* \text{ s.t. } f_3 (y) = x) \end{aligned}$$

Clearly, if the type is isomorphisms $V \to V^*$ is equivalent to isomorphisms $V^* \to V$. Hence, it suffices to show that $V^* \to V$ is equivalent to $V^* \times V^* \to \mathbb{R}$ to complete the equivalence. This is essentially identical to the above:

$$\begin{aligned} f_4 &= \lambda x : V^* \to V.\lambda (y, z) : V^* \times V^*.z (x (y)) \\ g_4 &= \lambda x : V^* \times V^* \to \mathbb{R}.\text{the} (y : V^* \to V \text{ s.t. } f_4 (y) = x) \end{aligned}$$

The idea behind both of these equivalences is the same: all three of these maps may, once we fix a basis, be represented as nonsingular matrices: if $f : V \to V^*$, then we may write $f(x) = Mx$; if $f : V \times V \to \mathbb{R}$, then we may write $f(x, y) = xMy$, and likewise if $f : V^* \times V^* \to \mathbb{R}$. Furthermore, in all of these cases, there is a $1-1$ correspondence between matrices, and functions of the desired form. In order to find an equivalence, then, we need only impose the condition that "the matrices are the same".

Problem 3

This will be a proof by contradiction. Suppose that V and V^* are equivalent. Then there must exist a well typed bijective $f : V \to V^*$. Let $\varphi : V \to V$ be an automorphism (an isomorphism between V and itself). By parametricity, $f \sim_\varphi f$, giving that $\forall x : V.\forall y : V.\, (f(x)(y) = f(\varphi(x))(\varphi(y)))$ (this holds because $f(x)(y) : \mathbb{R}$, and equivalence for real numbers is just equality).

In particular, if we assume that V is at least two dimensional, and let $v_i : i \in \{1, \ldots, d\}$ be a basis for V, then we may define an automorphism $\varphi : V \to V$ such that if $x = \sum_{i=1}^{d} \alpha_i v_i$ is the basis expansion for x, φ will exchange the coefficients α_1 and α_2: $\varphi(x) = \alpha_2 v_1 + \alpha_1 v_2 + \sum_{i=3}^{d} \alpha_i v_i$. By the fact that $f(x)(y) = f(\varphi(x))(\varphi(y))$, we will have that $f(v_1)(v_1) = f(v_2)(v_2)$ and $f(v_1)(v_2) = f(v_2)(v_1)$. Since every pair of linearly independent vectors may be extended into a basis (if V is at least two dimensional), we see that $f(x)(y) = f(y)(x)$ for all linearly independent $x, y : V$.

Now consider the automorphism $\varphi = \lambda x : V.\alpha x$ for some $\alpha : \mathbb{R}$ with $\alpha \neq 0$, which, again using the fact that $f(x)(y) = f(\varphi(x))(\varphi(y))$, gives that $f(x)(y) = f(\alpha x)(\alpha y) = \alpha f(\alpha x)(y)$ (the last step by linearity), so that $f(x) = \alpha f(\alpha x)$ for all $x : V$. Combining this result with the previous symmetry result gives that $f(x)(y) =$

$\alpha f(\alpha x)(y) = \alpha f(y)(\alpha x) = \alpha^2 f(y)(x) = \alpha^2 f(x)(y)$ for all nonzero $\alpha : \mathbb{R}$, assuming that x and y are linearly independent. Taking $\alpha = 2$ gives that $f(x)(y) = 4f(x)(y)$, showing that $f = 0$, contradicting our assumption that it is a bijection.

So we have proved that V and V^* are not equivalent for V of dimension 2 and higher. Clearly, if V is zero-dimensional, then V and V^* are equivalent, since both V and V^* consist of a single element (the zero vector). What about if V is one-dimensional? In this case, we must still have by the above argument that $f(x) = \alpha f(\alpha x)$, from which we may infer that the bijection $f : V \to V^*$ must look something like the function $\frac{1}{x}$, and indeed, if we choose $f = \lambda x : V.\text{the}\,(y : V^* \text{ s.t. } y(x) = 1_{\mathbb{R}})$, then we get a function of precisely this form for all $x \neq 0$. With some added handling for the zero vector, our bijection becomes:

$$
\begin{aligned}
f \quad = \quad & \lambda x : V.\text{the}\,(y : V^* \text{ s.t.} \\
& ((x = 0_V) \Rightarrow (y = 0_{V^*})) \wedge \\
& ((x \neq 0_V) \Rightarrow (y(x) = 1_{\mathbb{R}})))
\end{aligned}
$$

Suprisingly, this is a bijection, for one-dimensional V, because any nonzero linear function crosses 1 at a unique point, and knowledge of this point uniquely determined the function (as the unique line which passes through the origin, and this point).