TTIC 106: Problem Set 1
Due Wednesday, January 27

1. Give a derivation of the following sequent using the axioms in the first installment of the notes (A1 through A21).

$$\emptyset\ ;\ \text{even} = (\lambda(x : \mathbf{int})\ \exists y : \mathbf{int}\ \ x = y + y)\ \ \vdash\ \ \text{even} : \mathbf{int} \to \mathbf{Boole}$$

Your derivation should be a numbered sequence of lines where each line is justified by an axiom and a specification of which preceding lines are used as antecedents.

2. Define a function if : $\mathbf{Boole} \times \mathbf{int} \times \mathbf{int} \to \mathbf{int}$ with the property that $\text{if}(\Phi, \text{n}, \text{m})$ equals n if Φ is true and m otherwise. Your definition should be well formed under axioms A1 through A21. More specifically, it should be given as if $= e$ and the axioms should allow you to derive $\emptyset\ ;\ \text{if} = e \vdash \text{if} : \mathbf{Boole} \times \mathbf{int} \times \mathbf{int} \to \mathbf{int}$. You do not have to give the derivation.

3. Give a definition of the factorial function of the form fact $= e$ which is well formed under the axioms A1 through A21. We adopt the convention that $\text{fact}(n) = 1$ for $n \leq 0$. For your definition e give a derivation of the following sequent.

$$\emptyset\ ;\ \text{fact} = e \vdash \text{fact} : \mathbf{int} \to \mathbf{int}$$

4. Recursive definitions are usually phrased formally in terms of a fixed point operator as in the following definition of factorial.

$$\text{fact} = \text{fix}((\lambda\ (f : \mathbf{int} \to \mathbf{int})\ (\lambda\ (x : \mathbf{int})\ \text{if}(x \leq 1, 1, x * f(x - 1)))))$$

Give a well-formed definition of the function fix such that this definition of fact is both well formed and correct and also works for other "recursive definitions". More specifically, let F be the argument to fix in the above expression. Note that $F(\text{fact}) = \text{fact}$. The function fact is the unique fixed point of F. The function fix should return the fixed point of its argument F whenever F has a unique fixed point. You do not have to give the well-formedness derivation of your definition.