TTIC 106: Second Midterm Exam

Friday February 26

Recall that two types τ and σ are equivalent in context Σ if there exist expressions f and g such that we have the following.

$$
\begin{array}{lcl}
\Sigma & \vdash & f : \tau \rightarrow \sigma \\
\\
\Sigma & \vdash & g : \sigma \rightarrow \tau \\
\\
\Sigma & \models & \forall x : \tau \; g(f(x)) = x \\
\\
\Sigma & \models & \forall y : \sigma \; f(g(y)) = y
\end{array}
$$

1. Show that the structure type $\{\text{first} : \mathbf{int}\ ;\ \text{second} : \mathbf{real}\}$ is equivalent to the structure type $\{\text{first} : \mathbf{real}\ ;\ \text{second} : \mathbf{int}\}$. You can assume that we have the following simple rule for typing structure expressions.

Structure Term Formation (Simple Form):

$$
\frac{\begin{array}{l}\Sigma \vdash e_1 : \tau_1 \\ \vdots \\ \Sigma \vdash e_n : \tau_n\end{array}}{\Sigma \vdash \{s_1 \leftarrow e_1; \cdots ; s_n \leftarrow e_n\} : \{s_1 : \tau_1; \cdots ; s_n : \tau_n\}}
$$

2. Let vectors $x_1, \ldots x_d$ be a basis for a vector space V and let $f_1, \ldots f_d$ be a basis for the dual space V^*. We say that the dual basis corresponds to the primal basis if we have the following.

$$(f_1(x_i), \ldots, f_{i-1}(x_i), f_i(x_i), f_{i+1}(x_i), \ldots, f_d(x_i)) = (0, \ldots, 0, 1, 0, \ldots, 0)$$

or equivalently

$$f_j(x_i) = \left\{ \begin{array}{ll} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{array} \right.$$

It is a true fact that for any primal basis there is exactly one corresponding dual basis and for any dual basis there is exactly one corresponding primal basis. You do not have to prove this fact. However, given this fact argue that the type "basis for V" is equivalent to the type "basis for V^*".

3. Suppose that in the context Σ we can show that the types τ and σ are equivalent. Show that for any third type γ with $\Sigma \vdash \gamma : \mathbf{type}_i$ we have that the type $\sigma \to \gamma$ is equivalent to the type $\tau \to \gamma$ and that the type $\gamma \to \sigma$ is equivalent to the type $\gamma \to \tau$ (this is a kind of substitution of equivalents for equivalents in type expressions).

4. For which of the following triples of a context and two types is the first type equivalent to the second type in the given context.

context	type 1	type 2	equivalent	not equivalent
$\alpha : \mathbf{type}_1$	$\alpha \to (\alpha \to \mathbf{Boole})$	$\alpha \times \alpha \to \mathbf{Boole}$		
$\alpha : \mathbf{type}_1$	$(\alpha \to \alpha) \to \mathbf{Boole}$	$\alpha \times \alpha \to \mathbf{Boole}$		
$\alpha : \mathbf{type}_1$	$\mathbf{Boole} \to \alpha$	$\{\mathrm{first} : \alpha \,;\, \mathrm{second} : \alpha\}$		
$\alpha : \mathbf{type}_1$	$\{\mathrm{first} : \alpha \,;\, \mathrm{second} : \alpha\} \to \mathbf{Boole}$	$\alpha \times \alpha \to \mathbf{Boole}$		

5. For which of the following pairs of a context Σ and a type τ does there exists an expression e such that we have $\sigma \vdash e : \tau$

context	type	exists expression	not exists expression
$\alpha : \mathbf{type}_1$	α		
$\alpha : \mathbf{type}_1 \,;\, P : \alpha \to \mathbf{Boole}$	α		
$\alpha : \mathbf{type}_1 \,;\, f : \alpha \times \alpha \to \alpha$	$\alpha \to (\alpha \to \alpha)$		
$\alpha : \mathbf{type}_1 \,;\, f : \alpha \to \mathbf{int}$	α		
$\alpha : \mathbf{type}_1 \,;\, f : \alpha \to \mathbf{int}$	$\alpha \to \mathbf{Boole}$		