Low-distortion Embeddings of General Metrics Into the Line

Mihai Badoiu Julia Chuzhoy Piotr Indyk Anastasios Sidiropoulos

CSAIL, MIT

{mihai,cjulia,indyk,tasos}@mit.edu

Abstract

A low-distortion embedding between two metric spaces is a mapping which preserves the distances between each pair of points, up to a small factor called distortion. Low-distortion embeddings have recently found numerous applications in computer science.

Most of the known embedding results are "absolute", that is, of the form: any metric Y from a given class of metrics C can be embedded into a metric X with low distortion c. This is beneficial if one can guarantee low distortion for all metrics Y in C. However, in many situations, the worst-case distortion is too large to be meaningful. For example, if X is a line metric, then even very simple metrics (an n-point star or an n-point cycle) are embeddable into X only with distortion linear in n. Nevertheless, embeddings into the line (or into low-dimensional spaces) are important for many applications.

A solution to this issue is to consider "relative" (or "approximation") embedding problems, where the goal is to design an (a-approximation) algorithm which, given any metric X from C as an input, finds an embedding of X into Y which has distortion $a * c_Y(X)$, where $c_Y(X)$ is the best possible distortion of an embedding of X into Y.

In this paper we show algorithms and hardness results for relative embedding problems. In particular we give:

- an algorithm that, given a general metric M, finds an embedding with distortion $O(\Delta^{4/5}\text{poly}(c_{line}(M)))$, where Δ is the spread of M
- an algorithm that, given a weighted tree metric M, finds an embedding with distortion $\text{poly}(c_{line}(M))$
- a hardness result, showing that computing minimum line distortion is hard to approximate up to a factor polynomial in n, even for weighted tree metrics with spread $\Delta = n^{O(1)}$.

1 Introduction

A low-distortion embedding between two metric spaces with distance functions D and D' is a (non-contractive) mapping f such that for any pair of points p, q in the original metric, their distance $D(p, q)$ before the mapping is the same as the distance $D'(f(p), f(q))$ after the mapping, up to a (small) multiplicative factor c. Low-distortion embeddings have been a subject of extensive mathematical studies. More recently, they found numerous applications in computer science (cf. [Lin02, Ind01]).

Most of the research on embeddings focused on showing *absolute* results, of the form:

> Given a class of metrics C and a metric Y, what is the *smallest distortion* $c \geq 1$ such that any metric $X \in C$ can be embedded into Y with distortion c ?

Paper	From	Into	Distortion	Comments
[LLR94]	general metrics	l_2	c	uses SDP
[KRS04]	line	line	c	c is constant and embedding is a bijection
	unweighted graphs	bounded degree trees	c	as above
[EP04]	unweighted graphs	sub-trees	$O(c \log n)$	
[BIS04]	unweighted graphs	trees	$O(c)$	
[BDG$^+$05]	unweighted graphs	line	$O(c^2)$	implies $\sqrt{n}$-approximation
			$> ac$	Hard to a-approximate for some $a > 1$
			c	c is constant
	unweighted trees	line	$O(c^{3/2}\sqrt{\log c})$	
	subsets of a sphere	plane	$3c$	

Figure 1: Previous work on relative embedding problems for multiplicative distortion.

Very recently, a few papers addressed the *relative* [1] (or approximation) version of the problem, which is of the following form:

> Given a class of metrics C and a metric Y, what is the *smallest approximation factor* $a \geq 1$ of a polynomial-time algorithm minimizing the distortion of embedding of a given input metric $X \in C$ into Y ?

The relative formulation is of interest in situations where the absolute formulation yields distortion that is too large to be interesting or meaningful. A good example is the problem of embedding metrics into a line. Even simple metrics, such as an n-point star or an n-point cycle requires $\Omega(n)$ distortion when embedded into a line. Nevertheless, line embeddings, or, in general, embeddings into low-dimensional spaces, are important in many applications, such as visualisation (e.g., see [TdSL] or [MDS] web pages). Thus, it is important to design algorithms which produce low-distortion embeddings, if such embeddings are possible.

Despite the importance of the problem, not many relative embedding results are known. This is perhaps because the problems do not seem to be easily amenable[2] to standard approximation algorithms approaches (which were, e.g., successfully used for a closely related *bandwidth* problem [Fei00, DV01]). The results that we are aware of[3] are listed in Figure 1 (c denotes the optimal distortion, and n denotes metric size).

In this paper, we consider the problem of embedding metrics induced by *weighted* graphs into the line. The known algorithms were designed for *unweighted graphs* and thus provide only very weak guarantees for the problem. Specifically, assume that the minimum interpoint distance between the points is 1 and the maximum distance[4] is Δ. Then, by scaling, one can obtain algorithms for weighted graphs, with approximation factor multiplied by Δ.

Our results are presented in Figure 2. The first result is an algorithm that, given a general metric c-embeddable into the line, constructs an embedding with distortion $O(\Delta^{4/5}c^{13/5})$. The algorithm uses a

[1] The *absolute* and *relative* (resp.) versions of the problem were referred to as *combinatorial* and *algorithmic* (resp.) in [BDG$^+$05]. These terms could be confusing, however, since the *absolute* problem has both combinatorial and algorithmic components: in many applications it is important how to find low-distortion embeddings, in addition to knowing that such embeddings exist. Thus, to avoid misunderstanding, in this paper we use a different terminology.

[2] For example, there exist metrics for which any vertex ordering resulting in "low" bandwidth must result in "high" distortion when converted into a (non-contractive) embedding. This holds, e.g., for a metric induced by a "comb" graph, with a "teeth", each of length b, for $b >> a$. The row-by-row order, which minimizes the bandwidth, results in $\Omega(ab)$ distortion of the edges at the end of the teeth, while the column-by-column order gives distortion b.

[3] Note that the table contains only the results that hold for the *multiplicative* definition of the distortion. There is a rich body of work that applies to other definitions of distortion, notably the *additive* or *average* distortion, summarized in Section 1.1.

[4] We call the maximum/minimum interpoint distance ratio the *spread* of the metric.

From	Into	Distortion	Comments
general metrics	line	$O(\Delta^{4/5}c^{13/5})$	
weighted trees	line	$c^{O(1)}$	
weighted trees	line	$\Omega(n^{1/12}c)$	Hard to $O(n^{1/12})$-approximate even for $\Delta = n^{O(1)}$

Figure 2: Our results.

Paper	From	Into	Distortion	Comments
[FCKW93]	general distance matrix	ultrametrics	c	
[ABFC$^+$96]	general distance matrix	tree metrics	$3c$	
			$\geq 9/8c$	Hard to $9/8$-approximate
[HIL98]	general distance matrix	line	$2c$	
			$\geq 4/3c$	Hard to $4/3$-approximate
[Bŏ3]	general distance matrix	plane under l_1	$O(c)$	
[BDHI04]	general distance matrix	plane under l_2	$O(c)$	Time quasi-polynomial in Δ

Figure 3: Previous work on relative embedding problems for maximum additive distortion.

novel method for traversing a weighted graph. It also uses a modification of the unweighted-graph algorithm from [BDG$^+$05] as a subroutine, with a more general analysis.

Then, we consider the problem of embedding weighted tree metrics into the line. In this case we are able to get rid of the dependence on Δ from the approximation factor. Specifically, our algorithm produces an embedding with distortion $c^{O(1)}$.

We complement our upper bounds by a lower bound, which shows that the problem is hard to approximate up to a factor $a = \Omega(n^{1/12})$. This dramatically improves over the earlier result of [BDG$^+$05], which only showed that the problem is hard for some constant $a > 1$ (note however that their result applies to unweighted graph metrics as well). Since the instances used to show our hardness result have spread $\Delta \leq n^{O(1)}$, it follows that approximating the distortion up to a factor of $\Delta^{\Omega(1)}$ is hard as well. In fact, the instances used to show hardness are metrics induced by (weighted) *trees*; thus the problem is hard for tree metrics as well. Our hardness proof is inspired by the ideas of Unger [Ung98].

1.1 Related Work

Relative embedding problems have been theoretically studied for over a decade. Until recently, however, the research has been mostly focused on different notions of distortion. Specifically, several results gave been obtained for finding embedding f from space (X, D) into (X', D') that minimizes the *maximum additive distortion*, that is, minimizing $\max_{p,q \in X} |D(p, q) - D'(f(p), f(q))|$. The results are depicted in Figure 3. A few other results have been obtained for *average* distortion [Dha04, DGR04]; see the papers for results and problem definitions.

2 Preliminaries

Consider an embedding of a set of vertices V into the line. We say that $U \subset V$ is embedded *continuously*, if there are no vertices $x, x' \in U$, and $y \in V - U$, such that $f(x) < f(y) < f(x')$.

We say that vertex set U is embedded *inside* vertex set U' iff the smallest interval containing the embedding of U also contains the embedding of U'. In particular, we say that vertex v is embedded inside edge $e = (x, y)$ for $v \neq x, v \neq y$, if either $f(x) < f(v) < f(y)$ or $f(y) < f(v) < f(x)$ hold.

Let $M = (X, D)$ be a metric, and $f : X \to \mathbb{R}$ be a non-contracting embedding of M into the line. Then, the *length* of f is $\max_{u \in X} f(u) - \min_{v \in X} f(v)$.

3 General metrics

In this section we will present a polynomial-time algorithm that given a metric $M = (X, D)$ of spread Δ that c-embeds into the line, computes an embedding of M into the line, with distortion $O(c^{11/4}\Delta^{3/4})$. Since it is known [Mat90] that any n-point metric embeds into the line with distortion $O(n)$, we can assume that $\Delta = O(n^{4/3})$.

We view metric M as a complete graph G defined on vertex set X, where the weight of each edge $e = \{u, v\}$ is $D(u, v)$. As a first step, our algorithm partitions the point set X into sub-sets $X_1, \ldots, X_\ell$, as follows. Let W be a large integer to be specified later. Remove all the edges of weight greater than W from G, and denote the resulting connected components by $C_1, \ldots, C_\ell$. Then for each $i : 1 \leq i \leq \ell$, X_i is the set of vertices of C_i. Let G_i be the subgraph of G induced by X_i. Our algorithm computes a low-distortion embedding for each G_i separately, and then concatenates the embeddings to obtain the final embedding of M. In order for the concatenation to have small distortion, we need the length of the embedding of each component to be sufficiently small (relatively to W). The following simple lemma, essentially shown in [Mat90], gives an embedding that will be used as a subroutine.

Lemma 1. *Let $M = (X, D)$ be a metric with minimum distance 1, and let T be a spanning tree of M. Then we can compute in polynomial time an embedding of M into the line, with distortion $O(cost(T))$, and length $O(cost(T))$.*

The embedding in the lemma is computed by taking an (pre-order) walk of the tree T. Since each edge is traversed only a constant number of times, the total length and distortion of the embedding follows.

Our algorithm proceeds as follows. For each $i : 1 \leq i \leq \ell$, we compute a spanning tree T_i of G_i, that has the following properties: the cost of T_i is low, and there exists a walk on T_i that gives a small distortion embedding of G_i. We can then view the concatenation of the embeddings of the components as if it is obtained by a walk on a spanning tree T of G. We show that the cost of T is small, and thus the total length of the embedding of G is also small. Since the minimum distance between components is large, the inter-component distortion is small.

3.1 Embedding the Components

In this section we concentrate on some component G_i, and we show how to embed it into a line.

Let H be the graph on vertex set X_i, obtained by removing all the edges of length at least W from G_i, and let H' be the graph obtained by removing all the edges of length at least cW from G_i. For any pair of vertices $x, y \in X_i$, let $D_H(x, y)$ and $D_{H'}(x, y)$ be the shortest-path distances between x and y in H and H', respectively. Recall that by the definition of X_i, H is a connected graph, and observe that $D_H(x, y) \geq D_{H'}(x, y) \geq D(x, y)$.

Lemma 2. *For any $x, y \in X_i$, $D_{H'}(x, y) \leq cD(x, y)$.*

Proof: Let f be an optimal non-contracting embedding of G_i, with distortion at most c. Consider any pair u, v of vertices that are embedded consecutively in f. We start by showing that $D(u, v) \leq cW$. Let T be the minimum spanning tree of H. If edge $\{u, v\}$ belongs to T, then $D(u, v) \leq W$. Otherwise, since T is connected, there is an edge $e = \{u', v'\}$ in tree T, such that both u and v are embedded inside e. But then

$D(u', v') \leq W$, and since the embedding distortion is at most c, $|f(u) - f(v)| \leq |f(u') - f(v')| \leq cW$. As the embedding is non-contracting, $D(u, v) \leq cW$ must hold.

Consider now some pair $x, y \in X_i$ of vertices. If no vertex is embedded between x and y, then by the above argument, $D(x, y) \leq cW$, and thus the edge $\{x, y\}$ is in H' and $D_{H'}(x, y) = D(x, y)$. Otherwise, let $z_1, \ldots, z_k$ be the vertices appearing in the embedding f between x and y (in this order). Then the edges $\{x, z_1\}, \{z_1, z_2\}, \ldots, \{z_{k-1}, z_k\}, \{z_k, y\}$ all belong to H', and therefore

$$\begin{aligned} D_{H'}(x, y) &\leq D_{H'}(x, z_1) + D_{H'}(z_1, z_2) + \ldots D_{H'}(z_{k-1}, z_k) + D_{H'}(z_k, y) \\ &= D(x, z_1) + D(z_1, z_2) + \ldots D(z_{k-1}, z_k) + D(z_k, y) \\ &\leq |f(x) - f(z_1)| + |f(z_1) - f(z_2)| + \ldots + |f(z_{k-1}) - f(z_k)| + |f(z_k) - f(y)| \\ &= |f(x) - f(y)| \leq cD(x, y) \end{aligned}$$

□

We can now concentrate on embedding graph H'. Since the weight of each edge in graph H' is bounded by $O(cW)$, we can use a modified version of the algorithm of [BDG+05] to embed each G_i. First, we need the following technical Claim.

Claim 1. *There exists a shortest path $p = v_1, \ldots, v_k$, from u to u' in H', such that for any i, j, with $|i - j| > 1$, $D(v_i, v_j) = \Omega(W|i - j|)$.*

Proof: Pick an arbitrary shortest path, and repeat the following: while there exist consecutive vertices x_1, x_2, x_3 in p, with $D_{H'}(x_1, x_3) < cW$, remove x_2 from p, and add the edge $\{x_1, x_3\}$ in p. □

The algorithm works as follows. We start with the graph H', and we guess points u, u', such that there exists an optimal embedding of G_i having u and u' as the left-most and right-most point respectively. Let $p = (v_1, \ldots, v_k)$ be the shortest path from u to u' on H' (here $v_1 = u$ and $v_k = u'$), that is given by Claim 1. We partition X_i into clusters $V_1, \ldots, V_k$, as follows. Each vertex $x \in X_i$ belongs to cluster V_j, that minimizes $D(x, v_j)$.

Our next step is constructing super-clusters $U_1, \ldots, U_s$, where the partition induced by $\{V_j\}_{j=1}^{k}$ is a refinement of the partition induced by $\{U_j\}_{j=1}^{s}$, such that there is a small-cost spanning tree T' of G_i that "respects" the partition induced by $\{U_j\}_{j=1}^{s}$. More precisely, each edge of T' is either contained in a super-cluster U_i, or it is an edge of the path p. The final embedding of G_i is obtained by a walk on T', that traverses the super-clusters $U_1, \ldots, U_s$ in this order.

Note that there exist metrics over G_i for which any spanning tree that "respects" the partition induced by V_j's is much more expensive that the minimum spanning tree. Thus, we cannot simply use $U_j = V_j$.

We now show how to construct the super-clusters $U_1, \ldots, U_s$. We first need the following three technical claims, which constitute a natural extensions of similar claims from [BDG+05] to the weighted case. Their proofs are given in Appendix, Section A.

Claim 2. *For each $i : 1 \leq i \leq k$, $\max_{u \in V_i} \{D(u, v_i)\} \leq c^2 W/2$.*

Claim 3. *For each $r \geq 1$, and for each $i : 1 \leq i \leq k - r + 1$, $\sum_{j=i}^{i+r-1} |V_i| \leq c^2 W(c + r - 1) + 1$.*

Claim 4. *If $\{x, y\} \in E(H')$, where $x \in V_i$, and $y \in V_j$, then $D(v_i, v_j) \leq cW + c^2 W$, and $|i - j| = O(c^2)$.*

Let α be an integer with $0 \leq \alpha < c^4 W$. We partition the set X_i into super-clusters $U_1, \ldots, U_s$, such that for each $l : 1 \leq l \leq s$, U_l is the union of $c^4 W$ consecutive clusters V_j, where the indexes j are shifted by α. We refer to the above partition as *α-shifted*.

Claim 5. *Let T be an MST of G_i. We can compute in polynomial time a spanning tree T' of G_i, with cost$(T') = O($cost$(T))$, and an α-shifted partition of X_i, such that for any edge $\{x, y\}$ of T', either both $x, y \in U_l$ for some $l : 1 \leq l \leq s$, or $x = v_j$ and $y = v_{j+1}$ for some $j : 1 \leq j < k$.*

Proof: Observe that since H is connected, all the edges of T can have length at most W, and thus T is a subgraph of both H and H'. Consider the α-shifted partition obtained by picking $\alpha \in \{0, \ldots, c^4W - 1\}$, uniformly at random. Let T' be the spanning tree obtained from T as follows: For all edges $\{x, y\}$ of T, such that $x \in V_i \subseteq U_{i'}$, and $y \in V_j \subseteq U_{j'}$, where $i' \neq j'$, we remove $\{x, y\}$ from T, and we add the edges $\{x, v_i\}$, $\{y, v_j\}$, and the edges on the subpath of p from v_i to v_j. Finally, if the resulting graph T' contains cycles, we remove edges in an arbitrary order, until T' becomes a tree. Note that although T' is a spanning tree of G_i, it is not necessarily a subtree of H'.

Clearly, since the edges $\{x, v_i\}$, and $\{y, v_j\}$ that we add at each iteration of the above procedure are contained in the sets $U_{i'}$, and $U_{j'}$ respectively, it follows that T' satisfies the condition of the Claim.

We will next show that the expectation of cost(T'), taken over the random choice of α, is $O($cost$(T))$. For any edge $\{x, y\}$ that we remove from T, the cost of T' is increased by the sum of $D(x, v_i)$ and $D(y, v_j)$, plus the length of the shortest path from v_i to v_j in H'. Observe that the total increase of cost(T') due to the subpaths of p that we add, is at most cost(T). Thus, it suffices to bound the increase of cost(T') due to the edges $\{x, v_i\}$, and $\{y, v_j\}$.

By Claim 2, $D(x, v_i) \leq c^2W/2$, and $D(y, v_j) \leq c^2W/2$. Thus, for each edge $\{x, y\}$ that we remove from T, the cost of the resulting T' is increased by at most $O(c^2W)$.

For each i, the set $U_i \cup U_{i+1}$ contains $\Omega(c^4W)$ consecutive clusters V_j. Also, by Claim 4 the difference between the indexes of the clusters V_{t_1}, V_{t_2} containing the endpoints of an edge, is at most $|t_1 - t_2| = O(c^2)$. Thus, the probability that an edge of T is removed, is at most $O(\frac{1}{c^2W})$, and the expected total cost of the edges in $E(T') \setminus E(T)$ is $O(|X_i|) = O($cost$(T))$. Therefore, the expectation of cost(T'), is at most $O($cost$(T))$. The Claim follows by the linearity of expectation, and by the fact that there are only few choices for α. □

Let $U_1, \ldots, U_s$ be an α-shifted partition, satisfying the conditions of Claim 5, and let T' be the corresponding tree. Clearly, the subgraph $T'[U_i]$ induced by each U_i is a connected subtree of T'. For each U_i, we construct an embedding into the line by applying Lemma 1 on the spanning tree $T'[U_i]$. By Claim 3, $|U_i| = O(c^6W^2)$, and by Claim 2, the cost of the spanning tree $T'[U_i]$ of U_i is at most $O(|U_i|c^2W) = O(c^8W^3)$. Therefore, the embedding of each U_i, given by Lemma 1 has distortion $O(c^8W^3)$, and length $O(c^8W^3)$.

Finally, we construct an embedding for G_i by concatenating the embeddings computed for the sets $U_1, U_2, \ldots, U_s$, while leaving sufficient space between each consecutive pair of super-clusters, so that we satisfy non-contraction.

Lemma 3. *The above algorithm produces a non-contracting embedding of G_i with distortion $O(c^8W^3)$ and length $O($cost$($MST$(G_i)))$.*

Proof: Let g be the embedding produced by the algorithm. Clearly, g is non-contracting. Consider now a a pair of points $x, y \in X$, such that $x \in U_i$, and $y \in U_j$. If $|i - j| \leq 1$, then $|g(x) - g(y)| = O(c^8W^3)$, and thus the distortion of $D(x, y)$ is at most $O(c^8W^3)$.

Assume now that $|i - j| \geq 2$, and $x \in V_{i'}$, $y \in V_{j'}$. Then $|g(x) - g(y)| = O(|i - j| \cdot c^8W^3)$. On the other hand, $D(x, y) \geq D(v_{i'}, v_{j'}) - D(v_{i'}, x) - D(v_{j'}, y) \geq D(v_{i'}, v_{j'}) - c^2W \geq D_{H'}(v_{i'}, v_{j'})/c - c^2W \geq |i' - j'|/c - c^2W = \Omega(|i - j|c^4W^2)$. Thus, the distortion on $\{x, y\}$ is $O(c^7W^2)$. In total, the maximum distortion of the embedding g is $O(c^8W^3)$.

In order to bound the length of the constructed embedding, consider a walk on T' that visits the vertices of T according to their appearance in the line, from left to right. It is easy to see that this walk traverses each

edge at most 4 times. Thus, the length of the embedding, which is equal to the total length of the walk is at most $4\text{cost}(T') = O(\text{cost}(T))$. □

3.2 The Final Embedding

We are now ready to give a detailed description of the final algorithm. Assume that the minimum distance in M is 1, and the diameter is Δ. Let $H = (X, E)$ be a graph, such that an edge $(u, v) \in E$ iff $D(u, v) \leq W$, for a threshold W, to be determined later. We use the algorithm presented above to embed every connected component $G_1, \ldots, G_k$ of H. Let $f_1, f_2, \ldots, f_k$ be the embeddings that we get for the components $G_1, G_2, \ldots G_k$ using the above algorithm, and let T be a minimum spanning tree of G. It is easy to see that T connects the components G_i using exactly $k - 1$ edges.[5] We compute our final embedding f as follows. Fix an arbitrary Eulerian walk of T. Let P be the permutation of $(G_1, G_2, \ldots, G_k)$ that corresponds to the order of the first occurrence of any node of G_i in our traversal. Compute embedding f by concatenating the embeddings f_i of components G_i in the order of this permutation. Let T_i be the minimum spanning tree of G_i. Between every 2 consecutive embeddings in the permutation f_i and f_j, leave space $\max_{u \in G_i, v \in G_j} \{D(u, v)\} = D(a, b) + O(\text{cost}(T_i)) + O(cost(T_j))$, where $D(a, b)$ is the smallest distance between components G_i and G_j. This implies the next two Lemmata (see Appendix, Section A for proofs).

Lemma 4. *The length of f is at most $O(c\Delta)$.*

Lemma 5. *Let $a \in G_i, b \in G_j$ for $i \neq j$. Then $W \leq D(a, b) \leq |f(a) - f(b)| \leq O(c\Delta) \leq O(cD(a, b)\frac{\Delta}{W})$*

Theorem 1. *Let $M = (X, D)$ be a metric with spread Δ, that embeds into the line with distortion c. Then, we can compute in polynomial time an embedding of M into the line, of distortion $O(c^{11/4}\Delta^{3/4})$.*

Proof: Consider any pair of points. If they belong to different components, their distance distortion is $O(c\Delta/W)$ (Lemma 5). If they belong to the same component, their distance distortion is $O(c^8W^3)$ (Lemma 3). Setting $W = \Delta^{1/4}c^{-7/4}$ gives the claimed distortion bound. □

4 Hardness of Embedding Into the Line

In this section we show that even the problem of embedding weighted trees into the line is n^β-hard to approximate, for some constant $0 < \beta < 1$. Our reduction is performed from the 3SAT(5) problem, defined as follows. The input is a CNF formula φ, in which each clause consists of exactly 3 different literals and each variable participates in exactly 5 clauses, and the goal is to determine whether φ is satisfiable. Let $x_1, \ldots, x_n$, and $C_1, \ldots, C_m$, be the variables and the clauses of φ respectively, with $m = 5n/3$. Given an input formula φ, we construct a weighted tree G, such that if φ is satisfiable then there is an embedding of G into the line with distortion $O(b)$ (for some $b = \text{poly}(n)$) and if φ is not satisfiable, then the distortion of any embedding is at least $b\tau$, where $\tau = \text{poly}(n)$. The construction size is polynomial in τ, and hence the hardness result follows.

[5] Follows from correctness of Kruskal's algorithm. These $k - 1$ edges are exactly the last edges to be added because they are bigger than W and within components we have edges smaller than W

4.1 The construction

Our construction makes use of *caterpillar* graphs. A caterpillar graph consists of a path called *body*, and a collection of vertex disjoint paths, called *hairs*, while each hair is attached to a distinct vertex of the body, called the *base* of the hair. One of the endpoints of the caterpillar body is called the *first vertex* of the caterpillar, and the other endpoint is called the *last vertex*. We use two integer paremeters $b = \text{poly}(n)$ and $\tau = \text{poly}(n)$, whose exact value is determined later. We call a caterpillar graph a *canonical caterpillar*, if: (1) its body consists of integer-length edges, (2) the length of each hair is a multiple of b, and (3) each hair consists of edges of length $\frac{1}{b\tau}$. Our weighted tree G is a collection of canonical caterpillars, connected together in some way specified later. Notice that in any embedding of a canonical caterpillar with distortion less than $b\tau$, each hair must be embedded continuously (the formal proof appears below). Let $B_1, \ldots, B_t$ be caterpillars. A *concatenation* of $B_1, \ldots, B_t$ is a caterpillar obtained by connecting each pair of consecutive caterpillars B_i, B_{i+1} for $1 \leq i < t$ with a unit-length edge between the last vertex of B_i and the first vertex of B_{i+1}.

The building blocks of our graph G are literal caterpillars, variable caterpillars and clause caterpillars, that represent the literals, the variables and the clauses of the input formula φ. All these caterpillars are canonical. Let x_i be some variable in formula φ. We define two caterpillars called *literal caterpillars* w_i and w'_i, which represent the literals x_i and $\overline{x}_i$, respectively. Additionally, we have a *variable* caterpillar v_i representing variable x_i.

Let Y_L and Y_R be caterpillars whose bodies contain only one vertex (denoted by L and R respectively), with a hair of length $\tau^3 b$ (denoted by H_L and H_R respectively) attached to the body. The main part of our graph G is a canonical caterpillar W, defined as a concatenation of $Y_L, w_1, w'_1, w_2, w'_2, \ldots w_n, w'_n, Y_R$. The hairs of H_L and H_R are used as padding, to ensure that all the vertices of $G \setminus (H_L \cup H_R)$, are embedded between L and R. The length of the body of W is denoted by N, and is calculated later. Variable caterpillars v_i attach to W as follows. The first vertex of v_i connects by a unit-length edge to the first vertex of w'_i.

For every clause C_j in formula φ, our construction contains a canonical caterpillar k_j representing it, which is also called a *key*. Each key k_j is attached to vertex L by an edge of length N. Figure 5 (which appears in the Appendix) summarizes the above described construction.

We now provide the details on the structure of the literal caterpillars. Consider a literal ℓ, and let w be the caterpillar that represents it (i.e., if ℓ is x_i or $\overline{x}_i$, then w is w_i or $\overline{w}_i$). Assume that ℓ participates in (at most 5) clauses $C^\ell_1, C^\ell_2, \ldots$. Then w is the concatenation of at most 5 caterpillars, denoted by $h^\ell_1, h^\ell_2, \ldots$, that represent the participation of ℓ in these clauses (see Figure 4). Following [Ung98], we call these caterpillars *keyholes*. For convenience, we ensure that for each literal ℓ there are exactly 5 such keyholes $h^\ell_1, h^\ell_2, \ldots, h^\ell_5$, as follows. If the literal participates in less than 5 clauses, we use several copies of the same keyhole that corresponds to some clause in which ℓ participates. Thus, for each clause, for each literal participating in this clause, there is at least one keyhole. All the keyholes that correspond to the same clause C_j are copies of the same caterpillar $h(j)$, called *the keyhole* of C_j.

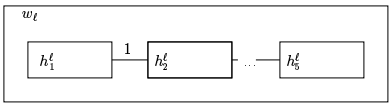


Figure 4: Caterpillar representing literal ℓ.

The main idea of the construction is as follows. First, the keys and the keyholes are designed in a special way, such that in order to avoid the distortion of $b\tau$, each key k_j has to be embedded inside one of the matching keyholes (copies of $h(j)$). The variable caterpillars are shaped in such a way that in any embedding with distortion less than $b\tau$, each variable caterpillar v_i is either embedded in w_i or w'_i. If v_i is embedded in w_i, then no key can be embedded inside any keyhole belonging to w_i without incurring the distortion of $b\tau$, and the same is true in case v_i is embedded into w'_i. Suppose formula φ is satisfiable. Then embedding of G with distortion $O(b)$ is obtained as

follows. We first embed hair H_L (starting from the vertex furthest from L), then the body of W and then H_R (starting from the vertex closest to R). For each variable x_i, if the correct assignment to x_i is TRUE, then variable caterpillar v_i is embedded inside the literal caterpillar w'_i, and otherwise it is embedded inside w_i. Given a clause C_j, if ℓ is the satisfied literal in this clause, we embed the key k_j in the copy of keyhole $h(j)$, that corresponds to literal ℓ. On the other hand, if φ is not satisfiable, we still need to embed each variable caterpillar v_i inside one of the two corresponding caterpillars w_i, w'_i, thus defining an assignment to all the variables. For example, if v_i is embedded inside w_i, this corresponds to the assignment FALSE to variable x_i. Such embedding of v_i will block all the keyholes in the caterpillar w_i. Since the assignment is non-satisfying, for at least one of the keys k_j, all the corresponding keyholes (copies of $h(j)$) are blocked, and so in order to embed k_j, we will need to incur a distortion of $b\tau$.

The rest of the construction description, including the implementation of keys and keyholes and variable caterpillars, as well as the reduction analysis, appears in Appendix, Section B.

5 Approximation Algorithm for Weighted Trees

In this section we consider embedding of weighted trees into the line. Given a weighted tree T, let φ be its optimal embedding into the line, whose distortion is denoted by c (we assume that $c \geq 200$). We provide a $\mathrm{poly}(c)$-approximation algorithm, which, combined with earlier work, implies $n^{1-\epsilon}$ approximation algorithm for weighted trees, for some constant $0 < \epsilon < 1$. The first step of our algorithm is guessing the optimal distortion c, and from now on we assume that we have guessed its value correctly.

We start with notation. Fix any vertex r of the tree to be the root. Given a vertex $v \neq r$, denote $d(v) = D(v, r)$. Consider any edge $e = (u, v)$. The length of e is denoted by w_e, and $d_e = \min\{d(u), d(v)\}$ is the distance of e from r. We say that e is a *large* edge if $w_e \geq \frac{d_e}{c}$, it is a *medium* edge if $\frac{d_e}{c} > w_e \geq \frac{d_e}{c^2}$, and otherwise e is a *small* edge.

Claim 6. *If $e = (u, v)$ is a medium or a small edge, then r is not embedded between u and v in the optimal solution.*

Proof: Assume otherwise. Then $|\varphi(u) - \varphi(v)| \geq d_e$. But $D(u, v) = w_e < \frac{d_e}{c}$, and edge e is stretched by a factor greater than c. □

Let $\mathcal{C}$ be the collection of connected components, obtained by removing all the large edges from the graph. For each component $C \in \mathcal{C}$, let $r(C)$ denote its "root", i.e. the vertex of C closest to r in tree T. We also denote by $e(C)$ the unique edge incident on $r(C)$ on the path from $r(C)$ to r, and by $\alpha(C)$ the length of this edge. Clearly, in the optimal solution, the embedding of component C lies completely to the left or to the right of r.

Given some component $C \in \mathcal{C}$, let $\ell(C)$ be the vertex in C that maximizes $D(r(C), \ell(C))$, and let $P(C)$ be the path between $r(C)$ and $\ell(C)$ in tree T. We define the *radius* of C to be $s(C) = D(r(C), \ell(C))$. Component C is called *large* if $s(C) > c^4\alpha(C)$, otherwise the component is called *small*. We define a tree T' of components, whose vertex set is $\mathcal{C} \cup \{r\}$, and the edges connecting the components are the same as in the original graph, (i.e., $e(C)$ for all $C \in \mathcal{C}$.)

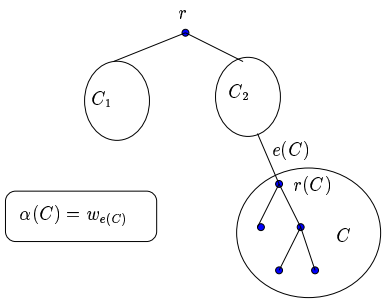
The main idea of our algorithm is to find the embedding of each one of the components separately recursively, and then concatenate these embeddings in some carefully chosen order. However, there is a problem with this algorithm, which is illustrated by the following example. Consider a large component C,

PROCEDURE PARTITION

Let $\mathcal{C}$ be the current set of all the components.
While there is a large component $C \in \mathcal{C}$, with a medium-sized edge e on the path from $r(C)$ to $\ell(C)$, such that the removal of e splits C into two large components, do:
Let C' and C'' be the two large components obtained by removing e. Remove C from $\mathcal{C}$ and add C' and C'' to $\mathcal{C}$.

consisting of a very long path, and a small component C' attached to this path in the middle. In this case any small-distortion embedding has to interleave the vertices of C and C', and thus our algorithm fails. We note that as $e(C')$ is a large edge, vertices of component C' have to be embedded into medium-sized edges of C (formal proof of this fact is provided later). In order to solve the above problem, we perform PROCEDURE PARTITION, that further subdivides large components by removing some medium-size edges from them.

From now on we only consider the components after the application of the above procedure, and the component graph, the values $r(C), \ell(C), \alpha(C)$ and so on are defined with respect to these components. It is easy to see that if a medium size edge e is incident on some component C, then C is a large component.



In fact, it is more convenient for us to define and solve a slightly more general problem. In the modified problem, in addition to a weighted tree T, we are also given a threshold value H. Given any embedding of our tree into the line, we say that it satisfies the *root condition* if: (1) each component C is embedded completely to the right or to the left of r, and (2) no component C with $\alpha(C) + s(C) \geq cH$ is embedded to the right of r. Our goal is to find an embedding that satisfies the root condition, while minimizing its distortion. Even though the problem might look artificial at this point, it is easy to see that by setting $H = \infty$, it converts to our original problem. The reason for defining the problem this way is that our algorithm solves the problem recursively on each component $C \in \mathcal{C}$, and then concatenates their embeddings into the final solution. In order to avoid large distortion of the distance between r and $r(C)$, we need to impose the root condition on the sub-problem corresponding to C with threshold $H = D(r, r(C))$. We later claim that for each sub-problem there is an optimal embedding with distortion c that satisfies the corresponding root condition.

5.1 The Structure of the Optimal Solution

In this section we explore some structural properties of the optimal solution, on which our algorithm relies.

Definition 1. *Let C, C' be two large components. We say that these components are* incompatible *if $s(C) > 2c^3\alpha(C')$ and $s(C') > 2c^3\alpha(C)$.*

The proof of the following lemma appears in Appendix.

Lemma 6. *If C and C' are large incompatible components, then in the optimal solution they are embedded on different sides of r.*

Definition 2. *Let C be a large component, and C' a small component. We say that there is a* conflict *between C and C' iff $2c^4\alpha(C) < \alpha(C') < s(C)/2c^4$.*

Lemma 7. *If C is a large component having a conflict with small component C', then C and C' are embedded on different sides of r in the optimal solution.*

The proof of the above lemma can be found in the Appendix.

Claim 7. *Let C, C' be large components and C'' a small component. Moreover, assume that there is a conflict between C and C'' and there is a conflict between C' and C''. Then C and C' are incompatible.*

Proof: Since there is a conflict between C and C'', $\alpha(C'') > 2c^4\alpha(C)$. A conflict between C' and C'' implies that $\alpha(C'') < s(C')/2c^4$. Therefore, $s(C') > 2c^3\alpha(C)$. Similarly, we can prove that $s(C) > 2c^3\alpha(C')$. □

We subdivide the small components into types or subsets $\mathcal{M}_1, \mathcal{M}_2, \ldots$. We say that a small component C is of *type* i and denote $C \in \mathcal{M}_i$ iff $c^{i-1} \leq \alpha(C) < c^i$.

Claim 8. *For each i, $|\mathcal{M}_i| \leq 4c^4$.*

Proof: Consider some $i \geq 1$, and assume that $|\mathcal{M}_i| > 4c^4$. Then in the optimal solution, there are more than $2c^4$ components of type i embedded on one of the sides of r. Denote these components by $C_1^i, C_2^i, \ldots, C_k^i$, $k > 2c^4$, and assume that vertices $r(C_j^i)$ are embedded in the optimal solution in this order, where $r(C_1^i)$ is embedded closest to r. It is easy to see that for any pair C, C' of small components, the distance between $r(C)$ and $r(C')$ is at least $\frac{\alpha(C)}{c}$. As the optimal embedding is non-contracting, for every $j = 1, \ldots, k-1$, there is a distance of at least $\alpha(C_j^i)/c \geq c^{i-2}$ between the embedding of $r(C_j^i)$ and $r(C_{j+1}^i)$. Therefore, $r(C_k^i)$ is embedded at a distance at least $kc^{i-2} > 2c^{i+2}$ from r. However, $d(r(C_k^i)) \leq \alpha(C_k^i) + c\alpha(C_k^i) \leq 2c^{i+1}$, and thus this distance is distorted by more than a factor of c in the optimal embedding. □

5.2 The Approximation Algorithm

Our algorithm consists of three major phases. In the first phase we compute the set $\mathcal{C}$ of components, after performing PROCEDURE PARTITION. In the second phase, we solve the problem recursively for each one of the components $C \in \mathcal{C}$, where the threshold for the root condition becomes $H = D(r(C), r)$. In the final phase, we combine the recursive solutions to produce the final embedding.

Claim 9. *For each recursive call to our algorithm, there is an embedding of the corresponding instance with distortion c, that satisfies the root condition.*

Proof: Let C be a component, and let C' be a component obtained after decomposing C. We consider the recursive call in C'. Since C is just a subtree of T, it embeds into the line with distortion c. Let f be such an embedding of C with distortion c. W.l.o.g., we can assume that $r(C')$ is embedded to the left of $r(C)$. It suffices to show that f satisfies the root condition in component C'.

Observe that for the recursive call in C', the threshold value is $H = D(r(C), r(C'))$. All the edges of C' as not large w.r.to $r(C)$, thus all the vertices of C' are embedded to the left of $r(C)$. Assume now that the root condition is not satisfied for C'. This implies that there exists a component C'' that is obtained after decomposing C', such that $\alpha(C'') + s(C'') \geq cH$, and such that C'' is embedded to the left of $r(C')$. Thus, $f(r(C')) < f(l(C'')) < f(r(C))$. It follows that $|f(r(C')) - f(r(C))| > |f(r(C')) - f(l(C''))| \geq D(r(C'), l(C'')) = \alpha(C'') + s(C'') \geq cH = cD(r(C'), r(C))$, a contradiction. □

The final embedding is produced as follows. First, partition the set $\mathcal{C}$ of components into two subsets $\mathcal{R}$, $\mathcal{L}$, containing the components to be embedded to the right and to the left of r, respectively. The partition

procedure is explained below. The components in $\mathcal{L}$ are then embedded to the left of r, while the embedding of each component is determined by the recursive procedure call, and the embeddings of different components do not overlap. The order of components is determined as follows. For each small component C, let $f(C) = \alpha(C)$, and for each large component C', let $f(C') = s(C')/2c^4$. The order of embedding is according to $f(C)$, where the component C with smallest $f(C)$ is embedded closest to the root r. The embedding of components in $\mathcal{R}$ is performed similarly, except that the embedding of each component is the mirror image of the embedding returned by the recursive procedure call (so that the root condition holds in the right direction). We put enough empty space between the embeddings of different components to ensure that the embedding is non-contracting. In the rest of this section we show how to partition $\mathcal{C}$ into the subsets $\mathcal{R}$ and $\mathcal{L}$.

We start with large components. We translate the problem into an instance of 2SAT, as follows. We have one variable $x(C)$ for each large cluster C. Embedding C to the left of r is equivalent to setting $x(C) = T$. If two components C and C' are incompatible, we ensure that variables $x(C)$ and $x(C')$ get different assignments, by adding clauses $x(C) \vee x(C')$ and $\overline{x(C)} \vee \overline{x(C')}$. Additionally, if $s(C) + \alpha(C) > cH$, then we ensure that C is not embedded to the right of r by adding a clause $x(C) \vee x(C)$. The optimal solution induces a satisfying assignment to the resulting 2SAT formula, and hence we can find a satisfying assignment in polynomial time. The clusters C with $x(C) = T$ are added to $\mathcal{L}$ and all other clusters are added to $\mathcal{R}$.

Consider now any small cluster C. If $s(C) + \alpha(C) > cH$, then we add C to $\mathcal{L}$. Otherwise, if $s(C) + \alpha(C) \leq cH$, then there is at most one large component C' that has conflict with C. If such a component C' exists, then we embed C on the side opposite to that where C' is embedded. Otherwise, C is embedded to the left of r. Clearly, in any embedding consistent with the above decision the root condition is satisfied.

The analysis of this phase of the algorithm appears in Section C.3 of the Appendix, together with the proof of the following theorem:

Theorem 2. *The algorithm produces a non-contracting embedding with distortion bounded by* $c^{O(1)}$.

References

[ABFC+96] R. Agarwala, V. Bafna, M. Farach-Colton, B. Narayanan, M. Paterson, and M. Thorup. On the approximability of numerical taxonomy: (fitting distances by tree metrics). *7th Symposium on Discrete Algorithms*, 1996.

[BDG+05] M. Bădoiu, K. Dhamdhere, A. Gupta, Y. Rabinovich, H. Raecke, R. Ravi, and A. Sidiropoulos. Approximation algorithms for low-distortion embeddings into low-dimensional spaces. *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, 2005.

[BDHI04] M. Bădoiu, E. Demaine, M. Hajiaghai, and P. Indyk. Embeddings with extra information. *Proceedings of the ACM Symposium on Computational Geometry*, 2004.

[BIS04] M. Bădoiu, P. Indyk, and A. Sidiropoulos. A constant-factor approximation algorithm for embedding unweighted graphs into trees. *AI Lab Technical Memo AIM-2004-015*, 2004.

[Bŏŏ3] M Bădoiu. Approximation algorithm for embedding metrics into a two-dimensional space. *14th Annual ACM-SIAM Symposium on Discrete Algorithms*, 2003.

[DGR04] K. Dhamdhere, A. Gupta, and R. Ravi. Approximating average distortion for embeddings into line. *Proceedings of the Symposium on Theoretical Aspects of Computer Science (STACS)*, 2004.

[Dha04] K. Dhamdhere. Approximating additive distortion of line embeddings. *Proceedings of RANDOM-APPROX*, 2004.

[DV01] J. Dunagan and S. Vempala. On euclidean embeddings and bandwidth minimization. *Proceedings of the 5th Workshop on Randomization and Approximation*, 2001.

[EP04] Y. Emek and D. Peleg. Approximating minimum max-stretch spanning trees on unweighted graphs. *Proceedings of the ACM-SIAM Symposium on Discrete Algorithms*, 2004.

[FCKW93] M. Farach-Colton, S. Kannan, and T. Warnow. A robust model for fi nding optimal evolutionary tree. *Annual ACM Symposium on Theory of Computing*, 1993.

[Fei00] U. Feige. Approximating the bandwidth via volume respecting embeddings. *Journal of Computer and System Sciences*, 60(3):510–539, 2000.

[HIL98] J. Hastad, L. Ivansson, and J. Lagergren. Fitting points on the real line and its application to rh mapping. *Lecture Notes in Computer Science*, 1461:465–467, 1998.

[Ind01] P. Indyk. Tutorial: Algorithmic applications of low-distortion geometric embeddings. *Annual Symposium on Foundations of Computer Science*, 2001.

[KRS04] C. Kenyon, Y. Rabani, and A. Sinclair. Low distortion maps between point sets. *Annual ACM Symposium on Theory of Computing*, 2004.

[Lin02] N. Linial. Finite metric spaces - combinatorics, geometry and algorithms. *Proceedings of the International Congress of Mathematicians III*, pages 573–586, 2002.

[LLR94] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. *Proceedings of 35th Annual IEEE Symposium on Foundations of Computer Science*, pages 577–591, 1994.

[Mat90] J. Matoušek. Bi-lipschitz embeddings into low-dimensional euclidean spaces. *Comment. Math. Univ. Carolinae*, 31:589–600, 1990.

[MDS] MDS: Working Group on Algorithms for Multidimensional Scaling. Algorithms for multidimensional scaling. DIMACS Web Page.

[TdSL] J. B. Tenenbaum, V. de Silva, and J. C. Langford. A global geometric framework for nonlinear dimensionality reduction. `http://isomap.stanford.edu/`.

[Ung98] W. Unger. The complexity of the approximation of the bandwidth problem. *Annual Symposium on Foundations of Computer Science*, 1998.

A General metrics

Claim 2 For each i, with $1 \leq i \leq k$, $\max_{u \in V_i} D(u, v_i) \leq c^2 W/2$.
Proof: Let $u \in V_i$. Consider the optimal embedding f. Since $f(v_1) = \min_{w \in X} f(w)$, and $f(v_k) = \max_{w \in X} f(w)$, it follows that there exists j, with $1 \leq j < k$, such that

$$\min\{f(v_j), f(v_{j+1})\} < f(u) < \max\{f(v_j), f(v_{j+1})\}.$$

Assume w.l.o.g., that $f(v_j) < f(u) < f(v_{j+1})$. We have $D(u, v_j) \geq D(u, v_i)$, since $u \in V_i$. Since f is non-contracting, we obtain $f(u) - f(v_j) \geq D(u, v_j) \geq D(u, v_i)$. Similarly, we have $f(v_{j+1}) - f(u) \geq D(u, v_i)$. Thus, $f(v_{j+1}) - f(v_j) \geq 2D(u, v_i)$. Since $\{v_j, v_{j+1}\} \in E(H')$, we have $D(v_j, v_{j+1}) \leq cW$. Thus, $c \geq \frac{f(v_{j+1}) - f(v_j)}{D(v_{j+1}, v_j)} \geq \frac{2D(u, v_i)}{cW}$. □

Claim 3 For each $r \geq 1$, and for each i, with $1 \leq i \leq k - r + 1$, $\sum_{j=i}^{i+r-1} |V_i| \leq c^2 W(c + r - 1) + 1$.
Proof: Let $A = \bigcup_{j=1}^{i+r-1} V_i$. Let $x = \text{argmin}_{u \in A} f(u)$, and $y = \text{argmax}_{u \in A} f(u)$. Let also $x \in V_i$, and $y \in V_j$. Clearly, $|f(v_i) - f(v_j)| \leq cD(v_i, v_j) \leq cD_{H'}(v_i, v_j) \leq c^2 W|i - j| \leq c^2 W(r-1)$. By Claim 2, we have $D(x, v_i) \leq c^2 W/2$, and $D(y, v_j) \leq c^2 W/2$. Thus, $|f(x) - f(v_i)| \leq cD(x, v_i) \leq c^3 W/2$, and similarly $|f(y) - f(v_j)| \leq c^3 W/2$. It follows that $|f(x) - f(y)| \leq |f(x) - f(v_i)| + |f(v_i) - f(v_j)| + |f(v_j) - f(y)| \leq c^3 W + c^2 W(r - 1)$. Note that by the choice of x, y, and since the minimum distance in M is 1, and f is non-contracting, we have $\sum_{j=i}^{i+r-1} |V_i| \leq |f(x) - f(y)| + 1$, and the assertion follows. □

Claim 4 If $\{x, y\} \in E(H')$, with $x \in V_i$, and $y \in V_j$, then $D(v_i, v_j) \leq cW + c^2 W$, and $|i - j| = O(c^2)$.
Proof: Since $\{x, y\} \in E(H')$, we have $D(x, y) \leq cW$. By Claim 2, we have $D(x, v_i) \leq c^2 W/2$, and $D(y, v_j) \leq c^2 W/2$. Thus, $D(v_i, v_j) \leq D(v_i, x) + D(x, y) + D(y, v_j) \leq cW + c^2 W$.
By Claim 1, we have $|i - j| = O(D(v_i, v_j)/W) = O(c^2)$. □

Lemma 4 The length of f is at most $O(c\Delta)$.
Proof: The length of f is the sum of the lengths of all f_i and the space that we leave between every 2 consecutive f_i, f_j's. Then, by Lemma 3, the length of f_i is $O(c \cdot \text{cost}(T_i))$. Thus, the sum of the lengths of all f_i's is $O(c \cdot \text{cost}(T))$. The total space that we leave between all pairs of consecutive embeddings f_i is $\text{cost}(T) + 2\sum_{i=1}^{k} O(\text{cost}(T_i)) = O(\text{cost}(T))$. Therefore the total length of the embedding f is $O(\text{cost}(T))$. At the same time, the cost of T is at most the length of the optimal embedding f, which is $O(c\Delta)$. The statement follows. □

Lemma 5 Let $a \in G_i$ and $b \in G_j$ for $i \neq j$. Then $D(a, b) \leq |f(a) - f(b)| \leq O(cD(a, b)\frac{\Delta}{W})$
Proof: The first part $D(a, b) \leq |f(a) - f(b)|$ is trivial by construction, since we left enough space between components G_i and G_j. Since a and b are in difference connected components, we have $D(a, b) > W$. Using Lemma 4 we have that $|f(a) - f(b)| = O(c\Delta) = O(c\Delta \frac{D(a,b)}{W}) = O(cD(a, b)\frac{\Delta}{W})$. □

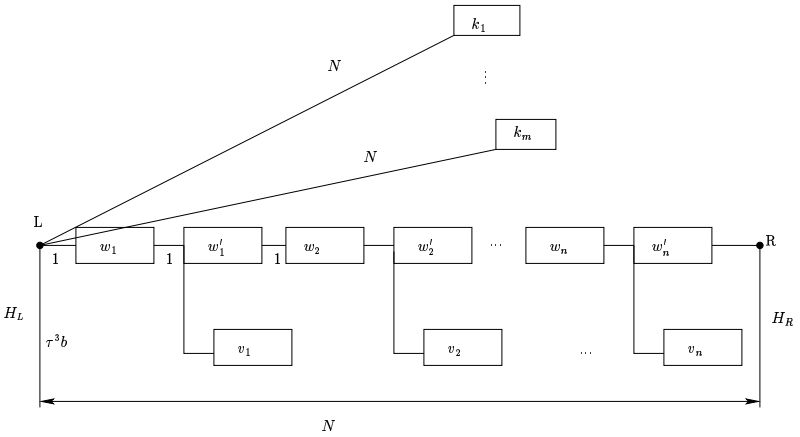


Figure 5: The high-level view of the construction.

B The Hardness Result

In this section we provide further details that complete the reduction description, followed by the reduction analysis. Note that all the caterpillars used in our construction are canonical, thus each hair of each caterpillar is a path of length-$\frac{1}{b\tau}$ edges.

B.1 Keys and Keyholes

We start with the following definition.

Definition 3. *For an integer α, a* barrier *caterpillar of length α consists of a body of α unit-length edges, and a hair of length b, attached to each one of the vertices of the body.*

Observe that the length of an embedding of a barrier of length α is at least αb. Intuitively, a barrier B of a "proper" length makes it impossible to embed a "short" edge (u, v) such that u and v are on the opposite sides of B, without incurring high distortion.

For a clause C_j, the corresponding keyhole $h(j)$ consists of three parts: *prefix*, *suffix* and the *main part*.

The prefix caterpillar, denoted by P, starts with a barrier of size τ^3, which is connected by an edge of length τ^2, called *large* edge, to vertex s which in turn is connected by a unit-length edge to a barrier of size $3\tau^4$. There is also a hair of length $b\tau^2$, called *large* hair, that attaches to vertex s.

The suffix caterpillar is denoted by S, and it is the mirror reflection of the prefix, where vertex s is denoted by t (see Figure 6).

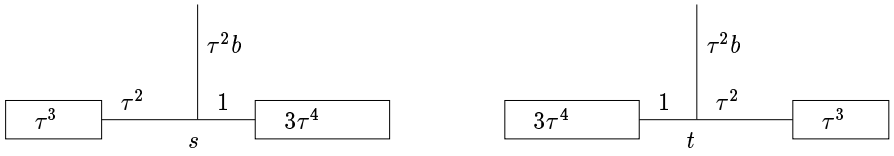


Figure 6: The prefix and the suffix.

The main part of keyhole $h(j)$ corresponding to clause C_j consists of m caterpillars $Q_1, Q_2, \ldots, Q_m$. Caterpillar Q_i, for $1 \leq i \leq j$ consists of a vertex z_i with a hair of length τb attached to it, which is referred

to as a *small hair*. Vertex z_i connects with an edge of length τ (called a *small edge*) to a barrier of size τ^2. For $j < i \leq m$, caterpillar Q_i is just a barrier of size τ^2. The keyhole h_j is defined to be the concatenation of $P, Q_1, \ldots, Q_m, S$.

We now proceed to define the keys. A key k_j is defined identically to the keyhole h_j, with the following changes:

- Observe that in the body of prefix P of $h(j)$, vertex s is adjacent to two edges, of sizes τ^2 and 1. We switch these two edges. We do the same with the two edges adjacent to vertex t in the body of suffix S. The resulting prefix and suffix are denoted by P' and S' respectively.
- Observe that each vertex z_i, $1 \leq i \leq j$ is attached in the body of $h(j)$ to two edges, of sizes 1 and τ. We switch these two edges.

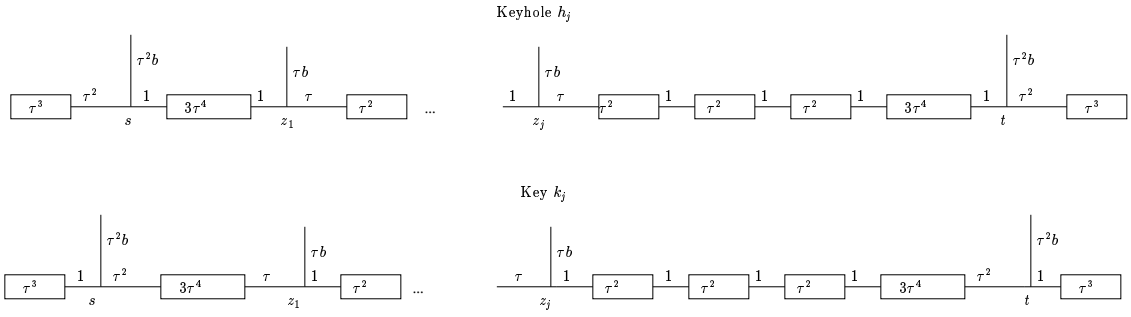


Figure 7: The key and the keyhole.

The intuition is that when any key is embedded into a keyhole, the two large hairs of the key have to be embedded inside the two large edges of the keyhole and vice versa, while the small hairs of both key and keyhole are embedded between the two long hairs. Similarly, the small hairs of the key have to be embedded inside the small edges of the keyhole and vice versa. Moreover, inside each small edge of a key (keyhole), at most one small hair of a keyhole (key) can be embedded, if the distortion is less than τb. Assume now that the key and the keyhole do not match, for example, we have key k_j and keyhole $h(i)$ where $j < i$. Then the number of small hairs in the keyhole is larger than the number of small edges in the key, and the distortion of embedding key k_j into keyhole $h(i)$ is large.

B.2 Variable caterpillars

We now define caterpillars v_i, representing variable x_i in formula φ.

Caterpillar v_i is a concatenation of five identical caterpillars $L_1, \ldots, L_5$. Caterpillar L_j for $1 \leq j \leq 5$ consists of three parts: The prefix P' and the suffix S' are identical to the prefix and the suffix of a key; the main part consists of m barriers of size τ^2 each, where each pair of consecutive barriers is connected by an edge of length τ.

The idea is that when v_i is embedded into w_i or w'_i, then each one of the caterpillars $L_1, \ldots, L_5$ will be embedded into the 5 corresponding keyholes, thus blocking them. More precisely, the 10 large hairs of v_i will be embedded into the 10 large edges of $L_1, \ldots, L_5$, ensuring that no large hair of any key can be embedded there.

B.3 Construction Size

We fix $\tau = n^{\mu}$ for some large integer μ. Our first step is bounding the length N of the body of W. Recall that W consists of $2n$ literal caterpillars, each consisting of 5 keyholes. The length of a keyhole is at most $m(\tau^2+\tau+1)+6\tau^4+2\tau^3+2\tau^2+2 < 7\tau^4$. Therefore, $N = O(\tau^4 n)$. We set $b = 3N$.

One can easily see that the size of the construction is dominated by the number of vertices on the hairs H_R and H_L. The length of each one of these hairs is $\tau^3 b$, and the length of each edge on a hair is $\frac{1}{b\tau}$. Therefore, the construction size is $O(\tau^4 b^2) = O(\tau^{12} n^2)$.

B.4 Analysis

In the following, we consider an embedding f of our graph G with distortion less than τb. We start by showing several structural properties of this embedding.

Claim 10. *Each hair of each caterpillar is embedded continuously.*

Proof: Assume otherwise. Then there is an edge $e = (x, y)$ on some hair H, and a vertex v not belonging to H embedded inside e. But the length of e is only $\frac{1}{\tau b}$, while the distance $D(x, v)$ is at least 1, and thus the distortion is at least τb. □

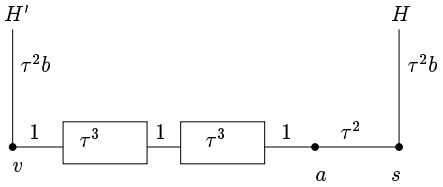
Claim 11. *The set of vertices in $G \setminus (H_L \cup H_R)$ is embedded continuously between the embeddings of L and R.*

Proof: By Claim 10, H_L and H_R are embedded continuously. Since the length of each H_L, and H_R is $\tau^3 b$, and the length of the longest edge of W is τ^2, it follows that $G \setminus (H_L \cup H_R)$ also has to be embedded continuously. Thus, in order to avoid distortion larger than τb, $G \setminus (H_L \cup H_R)$ has to be embedded between L and R. □

Our next goal is to prove that given some large edge $e = (u, v)$ on the body of W (which must belong to the prefix or the suffix of one of the keyholes), the only large hair of W that is embedded in it is the hair attached to u or v. The meaning of this claim is that the embedding of W has to be "nice", with the main part of each keyhole embedded between its prefix and suffix.

Claim 12. *Let h^i_j be any keyhole on caterpillar W, and let e be one of its large edges (assume w.l.o.g. that this edge is from its prefix). Let H be the large hair belonging to the prefix. Then H is the only hair belonging to W embedded inside e.*

Proof: We denote $e = (s, a)$, where s is the base of hair H. Recall that there is a barrier B_1 of size τ^3 attached to a. If h^i_j is not the first keyhole of W, then there is a suffix of another keyhole adjacent to B_1, with a barrier B_2 of size τ^3 attached to B_1 by a unit-length edge. The other endpoint of B_2 attaches by a unit-length edge to a base of a large hair H'. Clearly, H is embedded inside edge e continuously. Since the length of H is $\tau^2 b$, barriers B_1, B_2, and hair H' are embedded on the same side of H as vertex a.



Assume the claim is false, and let H'' be some other large hair belonging to some keyhole embedded inside e. Let x be the base of this hair. Since hair H'' is embedded inside edge e, so is its base x. Recall that vertex x attaches with a unit-length edge to a barrier B' of length $3\tau^4$. As the body of this barrier consists of unit-length edges, it has to be embedded completely between the embeddings of H and H'. The distance between s and the base of H' is only $2\tau^3 + \tau^2 + 3$, and thus the distance between their images in the embedding is at most $2\tau^4 b + \tau^2 b + 3b$. On the other hand, the size of the embedding of B' must be at least $3\tau^4 b$.

The only case we still need to consider is when h^i_j is the first keyhole on W. But then it is easy to see that the barrier B' has to be embedded between the embeddings of H and the hair H_L, which is again impossible.

□

The next corollary follows from Claim 12 and uses the fact that the main part of each keyhole only contains edges of length at most τ.

Corollary 1. *The main part of each keyhole is embedded between the two large hairs of the prefix and the suffix of the keyhole. Moreover, the large hairs of caterpillar W are embedded in the same order in which they appear on the body of W.*

Proof: Consider some keyhole h_j, and path P between s and t on its body. Recall that s and t serve as bases of large hairs whose length is $\tau^2 b$, and every edge on path P is of length at most τ. Therefore, all the vertices on path P and the hairs attached to them have to be embedded between the embeddings of these two large hairs.

Assume now that the large hairs on caterpillar W are not embedded in the same order in which they appear in W. Then there are three hairs H_1, H_2, H_3, such that H_1 and H_2 appear consecutively in W, but H_3 is embedded between H_1 and H_2. Let a and b be the bases of hairs H_1 and H_2. Then H_3 is embedded inside some edge e on the path (a, b). In order to avoid distortion τb, e has to be a large edge, and the only large edges between a and b are the two edges adjacent to a and b inside which the hairs H_1 and H_2 are embedded, which contradicts Claim 12 □

We prove next that for any large edge on any keyhole, at most one large hair of any key or a variable caterpillar can be embedded inside it.

Claim 13. *Let h_i be some keyhole, and let e be one of its large edges. Then there is at most one large hair belonging to any key or a variable caterpillar embedded inside e.*

Proof: Denote the endpoints of e by $\{v, u\}$. From the construction, there is a large hair H attached to one of these vertices, assume it's u. Recall also that both v and u are connected to barriers of size at least τ^3. Clearly, hair H is embedded inside e right next to vertex u. Suppose there are two other large hairs, H' and H'' embedded inside e, and assume that H'' is embedded between H and H'. Denote the base of the hair H'' by v''. Recall that v'' is connected by unit-length edge to a barrier of length τ^3. It is impossible to embed this whole barrier inside edge e, since the total length of such an embedding would be $\tau^3 b$, while the length of edge e is only τ^2. Therefore, there is at least one unit-length edge e' (part of the barrier body), whose one endpoint is embedded next to H'' and whose other endpoint is embedded outside e. But then one of the hairs H', H is embedded inside e', so it is impossible that the distortion is less than τb.

□

Using the same reasoning, we can prove the following two claims:

Claim 14. *For each small edge in a keyhole, only one small hair belonging to any key or a variable caterpillar can be embedded inside it.*

Claim 15. *For every key, for each one of its large (small, respectively) edges, at most one large (small, respectively) hair of a keyhole can be embedded inside it.*

Additionally, observe that the main part of any key k_i must be embedded completely between the prefix and the suffix of some keyhole h_j^ℓ and the large hairs of k_i are embedded into large hairs of h_j^ℓ. In this case we say that key k_i is embedded inside keyhole h_j^ℓ.

Yes instance

Note that the distance between any two vertices on the bodies of any caterpillars in our construction is at most $3N = b$.

Claim 16. *For each j, with $1 \leq j \leq m$, key k_j can be embedded inside a copy of $h(j)$ with distortion $O(b)$.*

Proof: The embedding is as follows. We move from left to right. While embedding the barriers, we embed a hair from the key and then a hair from the keyhole interchangeably, as follows: let H be a hair from the key and H' be a hair from a keyhole. We first embed H starting from its base, then we embed H' starting from the vertex furthest from its base. The distance between the embeddings of H and H' is $3b$, and thus the maximum stretch of an edge on the bodies of the barriers is $O(b)$. The large and the small hairs are embedded inside the large and the small edges respectively as follows. Let the endpoints of the large (small) edge of the key be denoted by v, u (the hair is attached to v), and denote the endpoints of the large (small) edge of the keyhole by u', v', the hair being attached to v'. We first embed vertex u', then the large (small) hair of the key, starting from v, then the large (small) hair of the keyhole (starting from the endpoint opposite to v', so v' is embedded last), and then vertex u. In case H, H' are large, the distance between their embeddings is $2\tau^2 b + b$, and if they are small, the distance is $2\tau b + b$. In any case, the distortion of this embedding is at most $O(b)$. □

For each variable caterpillar v_i, we can view its five sub-caterpillars $L_1, \ldots, L_5$ as "master keys" that can be embedded into any keyhole. We say that variable caterpillar v_i is embedded inside literal w iff the five sub-caterpillars of v_i are embedded into the five keyholes of w.

Similarly to Claim 16, we can prove the following claim.

Claim 17. *For each $i : 1 \leq i \leq n$, variable caterpillar v_i can be embedded inside each one of the literal caterpillars w_i or w_i' with distortion at most $O(b)$.*

Lemma 8. *If φ is satisfiable, then there exists an embedding of G into the line, with distortion at most $O(b)$.*

Proof: Consider the satisfying assignment to the variables, and assume the assignment to x_i is TRUE. Then, we embed v_i inside w_i'. Each clause contains at least one literal that satisfies it, so no variable caterpillar is embedded on this literal. We embed the key corresponding to the clause on the keyhole that belongs to that literal.

Finally, we embed H_L and H_R, to the left and to the right of the image of G, respectively. The maximum distortion of this embedding is at most $O(b)$. □

Unsatisfiable instance

Claim 18. *Suppose we have any embedding with distortion less than τb. Then each key is embedded in one of its corresponding keyholes.*

Proof: Suppose key k_i is embedded inside some keyhole h_j and $i \neq j$ (w.l.o.g., let $i < j$). Since all the small edges of k_i and the small hairs of h_j are embedded between the long hairs of k_i, and the number of small edges of k_i is less than the number of small hairs of h_j, the distortion must be at least $b\tau$. □

Claim 19. *Each variable caterpillar v_i is embedded inside either w_i, or w'_i. Moreover, once we embed v_i inside w_i or w'_i, it is impossible to embed any keys inside keyholes of w_i or w'_i, respectively, without incurring distortion τb.*

Proof: Let v_i be some variable caterpillar. Observe that there are 10 large hairs in v_i, which, in order to avoid distortion of τb, have to be embedded into 10 large edges of W. We prove that these have to be 10 consecutive large edges of w_i or of w'_i. Recall that the large hairs of W are embedded in the order in which they appear in W, each one of them is embedded into its adjacent large edge. The edge that attaches v_i to W is unit length, thus the first large hair of v_i has to be embedded into the hair of w'_i or w_i that lies closest to v_i. Observe also that large hairs of W can only be embedded inside large edges of v_i, and only one such hair is embedded into any large edge of v_i. Therefore, all the large hairs of v_i have to be embedded into the large edges of w_i or into the large edges of w'_i. Assume we embed v_i into w_i. Then inside each large edge of w_i, there is a large hair of v_i embedded in it. By Claim 13, it is impossible to embed additional large edge into this edge, thus none of the keys can be embedded into keyholes belonging to w_i. □

Lemma 9. *If φ is not satisfiable, then any embedding of G into the line has distortion at least τb.*

Proof: Assume we have an embedding with distortion less than τb. Then by Claim 18, each variable must be embedded in one of its corresponding literals, which implies an assignment to the variables. This assignment is not a satisfying one, so for some clause, for each one of its literals, there is a variable caterpillar embedded inside them, so it is impossible to embed the key corresponding to the clause into one of its keyholes, and the distortion must be at least τb. □

Theorem 3. *Given an M-point metric that c-embeds into the line, it is NP-hard to compute an embedding with distortion less than $\Omega(cM^{1/12-\epsilon})$ for arbitrarily small constant ϵ.*

Proof: Recall that our construction size is $M = \tau^{12}n^2$. If φ is satisfiable, then there is an embedding with distortion $O(b)$. Otherwise, any embedding has distortion at least τb. Since $\tau = n^{\mu}$ for a large enough constant μ, the theorem follows.

□

C Approximation Algorithm for Weighted Trees

In this section, we provide proofs omitted from Section 5.

C.1 Large Incompatible Components

The goal of this section is to prove Lemma 6

We start with the following claim:

Claim 20. *Let C and C' be two large incompatible components. Then in the optimal solution, vertex $\ell(C')$ is* **not** *embedded inside any edge of $P(C)$.*

Proof: Assume otherwise, and let $e = (u, v)$ be an edge of $P(C)$, with $d(u) < d(v)$, such that $\ell(C')$ is embedded between u and v. In order to finish our proof, it is enough to show that $D(u, \ell(C')) \geq d(u)$: in this case, if $\ell(C')$ is embedded between u and v, then $|\varphi(u) - \varphi(v)| \geq d(u)$, and as e is not a large edge, it is stretched by a factor greater than c in this embedding. It now only remains to prove that $D(u, \ell(C')) \geq d(u)$. For the sake of convenience, we denote $\ell = \ell(C')$.

We consider three cases. The first case is when the components C and C' are not the ancestor and descendant of one another in the tree of components. Let a be the least common ancestor of u and ℓ, note that $a \neq u, a \neq \ell$. Then $D(u, \ell) = D(a, u) + D(a, \ell)$. However, $D(a, \ell) \geq s(C') \geq c^4\alpha(C') \geq c^2 d_{e(C')} \geq d(a)$ (we are using the facts that C' is a large component and so $s(C') \geq c^4\alpha(C')$ and also that $e(C)$ is a large or a medium size edge, and therefore $\alpha(C') = w_{e(C')} \geq \frac{d_{e(C')}}{c^2}$). Thus, $D(u, \ell) \geq D(a, u) + d(a) \geq d(u)$ as desired.

The second case is when C' is a descendant of C in the tree of components. Let $a \in C$ be the least common ancestor of u and ℓ, note that $a = u$ is possible. Then $D(u, \ell) = D(u, a) + D(a, \ell)$. Again, $D(a, \ell) \geq s(C') \geq c^4\alpha(C') \geq c^2 d_{e(C')} \geq d(a)$ holds, and thus $D(u, \ell) \geq D(a, u) + d(a) \geq d(u)$.

The third case is when C' is an ancestor of C in the component tree. Let $a \in C'$ be the least common ancestor of u and ℓ. Notice first that $D(a, r(C')) < s(C')/2$ must hold, since otherwise $d_{e(C)} \geq D(a, r(C')) \geq s(C')/2 > c^3\alpha(C) = c^3 w_{e(C)}$, which is impossible since $e(C)$ is a large or a medium size edge. Assume now that $D(a, r(C')) < s(C')/2$ holds. But then $D(a, \ell) \geq s(C')/2 \geq c^3\alpha(C)$. To finish the proof, observe that $D(u, \ell) = D(a, \ell) + D(a, u) \geq c^3\alpha(C) + D(u, r(C)) \geq d(r(C)) + D(u, r(C)) \geq d(u)$. □

Lemma 10 (Lemma 6). *If C and C' are large incompatible components, then in the optimal solution they are embedded on different sides of r.*

Proof: Assume C and C' are embedded on the same side of r. As Claim 20 holds in both directions, the only way for C and C' to be embedded on the same side of r is when $\ell(C)$ is embedded between $r(C')$ and r or when $\ell(C')$ is embedded between $r(C)$ and r.

Assume w.l.o.g. that $\ell(C)$ is embedded between $r(C')$ and r. Since $D(\ell(C), r) \geq s(C) \geq 2c^3\alpha(C')$, vertices $r(C')$ and r are embedded at a distance at least $2c^3\alpha(C')$ from one another. However, $d(r(C')) = \alpha(C') + d_{e(C')} \leq \alpha(C') + c^2\alpha(C') < 2c^2\alpha(C')$ and thus this distance is distorted by more than a factor of c.

□

C.2 Combining Large and Small Components

This section is devoted to proving Lemma 7.

Lemma 11 (Lemma 7). *If C is a large component having a conflict with small component C', then C and C' are embedded on different sides of r in the optimal solution.*

Proof: Our proof consists of three claims. In the first claim, we show that if C and C' are embedded on the same side of r, then $r(C')$ is embedded inside some edge e on path $P(C)$. The second claim shows that C' must be a descendant of C in the tree of components. Finally, in the third claim, we show that edge e on path $P(C)$ into which $r(C')$ is embedded is a medium-size edge, whose removal splits C into two large components, therefore e must have been removed by PROCEDURE PARTITION.

Claim 21. *Assume that C and C' are embedded on the same side of r. Then $r(C')$ is embedded inside some edge e on path $P(C)$.*

Proof: Assume otherwise. Then either $r(C')$ is embedded between r and $r(C)$, or all the vertices on path $P(C)$ are embedded between $r(C')$ and r. If the former case is true, then $|\varphi(r) - \varphi(r(C))| > d(r(C')) \geq \alpha(C') \geq 2c^4\alpha(C)$. But $d(r(C)) = \alpha(C) + d_{e(C)} \leq \alpha(C) + c^2\alpha(C) < 2c^2\alpha(C)$. Thus, the distance between r and $r(C)$ is distorted by a factor greater than c.

If the latter is true, then $|\varphi(r) - \varphi(r(C'))| > s(C) > 2c^4\alpha(C')$. However, this means that the distance between r and $r(C')$ is distorted by a factor greater than c, since $d(r(C')) = \alpha(C') + d_{e(C')} \leq \alpha(C') + c\alpha(C') \leq 2c\alpha(C')$. □

Let $e = (u, v)$ denote the edge on path $P(C)$, such that $r(C')$ is embedded inside e, and assume w.l.o.g. that $d(u) < d(v)$.

Claim 22. *C' is a descendant of C in the tree of components.*

Proof: Assume otherwise. There are two cases to consider. If C is the descendant of C', then $d_{e(C)} \geq \alpha(C') \geq 2c^4\alpha(C)$, which is impossible since $e(C)$ is a large or a medium size edge.

The second case is when C and C' are not an ancestor-descendant pair. Let a be the least common ancestor of u and $r(C')$, and notice that $a \notin C'$. We show that $D(u, r(C')) \geq d(u)$, and thus $|\varphi(u) - \varphi(v)| \geq d(u)$ must hold, while $D(u, v) = w_e < d(u)/c$ since e is not large. Therefore, edge e is stretched by a factor greater than c, leading to a contradiction. To see that $D(u, r(C')) \geq d(u)$, Observe that $D(u, r(C')) \geq \alpha(C') + \alpha(C) + D(u, r(C))$. However, $\alpha(C') \geq 2c^4\alpha(C) \geq d_{e(C)}$ (we used the facts that C' and C have a conflict, and also that $e(C)$ is a large or a medium size edge). Therefore, $D(u, r(C')) \geq d(e(C)) + \alpha(C) + D(u, r(C)) \geq d(u)$.

□

Claim 23. *Edge e is of medium size, and upon its removal component C splits into two large components.*

Proof: We first show that e is a medium size edge. Let a be the least common ancestor of $r(C')$ and u. Since C' is a descendant of C, $a \in C$. Then $D(u, r(C')) = D(u, a) + D(a, r(C'))$. However, $D(a, r(C')) \geq \alpha(C') \geq \frac{d(a)}{c}$, since $e(C')$ is a large edge, and a is on the path from $r(C')$ to the root. Altogether, we have that $D(u, r(C')) \geq D(u, a) + \frac{d(a)}{c} \geq \frac{d(u)}{c}$. Since $r(C')$ is embedded between u and v, $|\varphi(u) - \varphi(v)| \geq \frac{d(u)}{c}$, and thus $D(u, v) = w_e \geq \frac{d(u)}{c^2}$ must hold.

Consider now two components C_1, C_2 obtained from C by removing edge e, and assume w.l.o.g. that $r(C) \in C_1$. We show that both these components are large.

Assume for contradiction that C_1 is small. On one hand, since C and C' have a conflict, $2c^4\alpha(C) < \alpha(C')$. On the other hand, since $r(C')$ is embedded inside edge e, and $D(u, r(C')) \geq \alpha(C')$, then $\alpha(C') \leq cw_e$ must hold. Combining the two inequalities together, we have: $2c^3\alpha(C) < w_e$. But since e is not large, $d(u) = d_e > w_e \cdot c > 2c^4\alpha(C)$. Finally, recall that $d(u) \leq D(u, r(C)) + \alpha(C) + c^2\alpha(C)$, and thus $D(u, r(C)) > c^4\alpha(C)$ must hold. But $D(u, r(C)) \leq s(C_1)$, and thus C_1 is a large component.

We now prove that C_2 is a large component. The main part of the proof is showing that $d(u) \leq \left(1 - \frac{1}{c}\right)\frac{s(C)}{c^3}$. Assume that the above bound is true. Then since e is not large, $w_e < \frac{d(u)}{c} \leq \left(1 - \frac{1}{c}\right)\frac{s(C)}{c^4}$. On the other hand, we can show that $s(C_2)$ is suffi ciently large relatively to w_e, as follows:

$$s(C_2) \geq s(C) - d(u) - w_e \geq s(C) - \left(1 - \frac{1}{c}\right)\frac{s(C)}{c^3} - \left(1 - \frac{1}{c}\right)\frac{s(C)}{c^4} \geq \left(1 - \frac{1}{c}\right)s(C)$$

Therefore, $s(C_2) \geq c^4 w_e$ holds, and C_2 is a large component.

It now only remains to prove that $d(u) \leq \left(1 - \frac{1}{c}\right)\frac{s(C)}{c^3}$. Recall that $r(C')$ is embedded between u and v, and thus the distance between the embeddings of u and v is at least:

$$D(u, r(C')) + D(v, r(C')) \geq 2D(u, r(C')) = 2[D(u, a) + D(a, r(C'))]$$

As the distortion is at most c,

$$w_e \geq 2\frac{D(u, a) + D(a, r(C'))}{c}$$

must hold. On the other hand, edge e is not large, and thus

$$w_e < \frac{d(u)}{c} = \frac{d(a) + D(u, a)}{c}$$

Combining the two inequalities together, we get:

$$d(a) \geq D(u, a) + 2D(a, r(C')) \geq D(u, a) + 2\alpha(C')$$

Since a is on the path from $r(C')$ to r and $e(C')$ is a large edge, $\alpha(C') \geq \frac{d(a)}{c}$. We thus have: $d(a)\left(1 - \frac{2}{c}\right) \geq D(u, a)$.

Altogether,

$$d(u) = d(a) + D(u, a) \leq d(a)\left(2 - \frac{2}{c}\right) \leq c\alpha(C')\left(2 - \frac{2}{c}\right) \leq \frac{s(C)}{c^3}\left(1 - \frac{1}{c}\right)$$

□

□

C.3 Analysis of the Algorithm

We start with the following simple observation.

Observation 1. *Let C be any component, and let r be the root of the current instance. Then $D(r(C), r) \leq 2c^2\alpha(C)$.*

Proof: It is easy to see that $D(r(C), r) = \alpha(C) + d_{e(C)}$. However, since $e(C)$ is a large or a medium size edge, $\alpha(C) \geq \frac{d_{e(C)}}{c^2}$. In total, $D(r(C), r) \leq \alpha(C) + c^2\alpha(C) \leq 2c^2\alpha(C)$. □

We now bound the empty space we need to leave between each pair of components that are embedded next to each other. Consider some component C embedded to the left of r. Recall that in the recursive procedure call for C, we use threshold value $H = D(r(C), r)$ for the root condition. Let $v \in C$ be the rightmost vertex in the embedding of C.

We want to show $D(v, r)$ is "small".Assume w.l.o.g. that $v \neq r(C)$. Let C' be the component, obtained by the decomposition of C, that contains v. Note that due to Observation 1, $D(r(C'), r(C)) \leq 2c^2\alpha(C')$. Since v (and therefore C') lies on the right side of $r(C)$, it must satisfy the threshold condition $\alpha(C') + s(C') \leq cH = cD(r(C), r)$. We can now write

$$\begin{aligned} D(v,r) &\leq D(r(C),r) + [D(v, r(C')) + D(r(C'), r(C))] \\ &\leq D(r(C),r) + [s(C') + 2c^2\alpha(C')] \\ &\leq D(r(C),r) + 2c^3 H \\ &\leq 3c^3 D(r(C),r) \\ &\leq 6c^5\alpha(C) \end{aligned}$$

For each component C embedded to the left of r, we leave empty space of $6c^5\alpha(C)$ to the right of the embedding of C, and empty space of $s(C) + D(r, r(C)) \leq s(C) + 2c^2\alpha(C)$ to the left of the embedding of C, such that empty spaces belonging to different components do not overlap. The embedding of components in $\mathcal{R}$ is performed similarly. It is easy to see that the resulting embedding is non-contracting.

Consider now some component C. Let $\mathcal{L}(C), \mathcal{S}(C)$ denote the sets of large and small components, respectively, embedded between C and r by our algorithm. We define $L(C) = \sum_{C' \in \mathcal{L}(C)} s(C')$, and $S(C) = \sum_{C' \in \mathcal{S}(C)} \alpha(C')$. In order to bound the approximation ratio of our algorithm, it is helpful to bound first the values $L(C)$ and $S(C)$ in terms of $\alpha(C)$.

Lemma 12. *For any component C, $L(C) \leq 4c^4\alpha(C)$, and $S(C) \leq 24c^8\alpha(C)$.*

Proof:

We start by bounding $L(C)$. Consider any pair C_1, C_2 of large components, embedded on the same side of r. These components are compatible, and thus we can assume w.l.o.g. that $s(C_1) \leq 2c^3\alpha(C_2)$. However, since C_2 is large, $\alpha(C_2) \leq s(C_2)/c^4$, and therefore $s(C_1) \leq 2s(C_2)/c$, and C_1 is embedded closer than C_2 to the root.

Assume now that C is a large component, and let $C' \in \mathcal{L}(C)$ be the component that maximizes $s(C')$. Then $s(C') \leq 2c^3\alpha(C)$ (since otherwise C must be embedded closer to r than C'). Moreover, the values of $s(C'')$ for $C'' \in \mathcal{L}(C)$ constitute a geometric series with ratio $\frac{2}{c}$. Therefore, $L(C) \leq 4c^3\alpha(C)$.

If C is a small component, let $C' \in \mathcal{L}(C)$ be the component that maximizes $s(C')$. Due to the ordering of the components by our algorithm, $\frac{s(C')}{2c^4} \leq \alpha(C)$. The values of $s(C'')$ for $C'' \in \mathcal{L}(C)$ again form a geometric series, and thus $L(C) \leq 4c^4\alpha(C)$.

We now proceed to bound $S(C)$. Recall that there are at most $4c^4$ small components of each type. Assume first that C is a small component of type i. Then $S(C)$ contains at most $4c^4$ components of the same type (whose α is less than $\alpha(C)$), and at most $4c^4$ components for each one of the types $1, \ldots, i-1$. Thus, $S(C) \leq 12c^4\alpha(C)$.

Suppose now that C is a large component, and let $C' \in \mathcal{S}(C)$ be the component maximizing $\alpha(C')$. Then $\alpha(C') \leq \frac{s(C)}{2c^4}$. Since there is no conflict between C and C', $\alpha(C') \leq 2c^4\alpha(C)$ must hold. Again, we have at most $4c^4$ components of the same type as C', whose α-value is not greater than $\alpha(C')$, and at most $4c^4$ components of each one of the smaller types. Therefore, $S(C) \leq 12c^4 \cdot 2c^4\alpha(C) \leq 24c^8\alpha(C)$. □

Definition 4. *Given a component C, its weight $W(C)$ is defined to be the sum of weights of its edges.*

Claim 24. $W(C) \leq 2cs(C)$

Proof: The length of any embedding of C is at least $W(C)$, while the maximum distance between any pair of points in C is $2s(C)$. Since the distortion of the optimal embedding is c, the claim holds. □

The next theorem is the central theorem in the analysis of our algorithm.

Theorem 4. *Let C be the instance of our problem with threshold H for the root condition. Then the algorithm produces an embedding with the following properties:*

- *The length of the embedding is at most $c^{13}W(C)$.*
- *The length of the embedding to the right of the root r is at most $c^{28}H$.*
- *For any vertex $v \in C$, v is embedded within distance $c^{29}D(v,r)$ from r.*

Proof:

The proof is by induction on the instance size. Let $\mathcal{C}$ be the collection of components produced by our algorithm. We assume that the claim holds for each $C' \in \mathcal{C}$ and the corresponding threshold value, and prove it for C.

We start by bounding the embedding length. We first bound the length of the embedding to the left of r. Let C_L be the leftmost component embedded to the left of r (if such a component exists). The length of the embedding to the left of r consists of the following parts: (1) the lengths of the embeddings of all the components in $\mathcal{L}$: they are bounded by $c^{13}\sum_{C' \in \mathcal{L}} W(C')$ by the inductive hypothesis; (2) the additional space we need to leave between the components to ensure non-contraction.

We show that this additional space is at most $c^{13}\alpha(C_L)$. Observe that edge $e(C_L)$ does not participate in any of the recursive algorithm executions. Since we can bound the length of the embedding to the right of r in a similar fashion, this will finish the proof that the total length of the embedding is at most $c^{13}W(C)$.

We now bound the additional space we need to place between the components. Let $C' \in \mathcal{L} \setminus \{C_L\}$ be some large component. The empty space we need to leave due to C' is at most $2[s(C') + D(r(C'), r)] \leq 2[s(C') + 2c^2\alpha(C')] \leq 3s(C')$ (since C' is large). Thus, in total, the large components in $\mathcal{L}\setminus\{C_L\}$ contribute at most $3L(C_L) \leq 12c^4\alpha(C_L)$. Consider now some small component $C' \in \mathcal{L} \setminus \{C_L\}$. The empty space due to C' is again bounded by $2[s(C') + D(r(C'), r)] \leq 2[s(C') + 2c^2\alpha(C')]$. However, since C' is small, $s(C') \leq c^4\alpha(C')$, and thus its contribution is at most $3c^4\alpha(C')$. In total, small components in $\mathcal{L} \setminus \{C_L\}$ contribute at most $3c^4S(C_L) \leq 72c^{12}\alpha(C_L)$. Finally, component C_L itself contributes at most $6c^5\alpha(C_L)$. The total additional empty space is thus at most:

$$12c^4\alpha(C_L) + 72c^{12}\alpha(C_L) + 6c^5\alpha(C_L) \leq c^{13}\alpha(C_L)$$

We now prove the second part of the theorem.

Let C_R be the rightmost component in our embedding. From the root condition, $\alpha(C_R) + s(C_R) \leq cH$. If C' is a large component embedded between C_R and r, then its embedding length is at most $c^{13}W(C') \leq$

$2c^{14}s(C')$. The amount of empty space we need to leave out for this component is at most $2[s(C') + D(r(C'), r)] \leq 2[s(C') + 2c^2\alpha(C')] \leq 3s(C')$. Thus, the total contribution of such components is at most $6c^{14}L(C_R) \leq 3c^{14} \cdot 4c^4\alpha(C_R) = 12c^{18}\alpha(C_R)$.

Similarly, the length of the embedding of a small component C' is at most $2c^{14}s(C') \leq 2c^{18}s(C')$, and the amount of free space we need to add due to C' is bounded by $2[s(C') + D(r(C'), r)] \leq 2[s(C') + 2c^2\alpha(C')] \leq 3c^4\alpha(C')$.The total contribution of small components is at most $3c^{18}S(C_R) \leq 3c^{18} \cdot 24c^8\alpha(C_R) \leq 72c^{26}\alpha(C_R)$. Finally, the length of the embedding of C_R is at most $2c^{14}s(C_R)$, and the empty space we need to leave to the left of it is at most $6c^5\alpha(C_R)$. The total size of the embedding to the right of r is at most:

$$12c^{18}\alpha(C_R) + 72c^{26}\alpha(C_R) + 6c^5\alpha(C_R) + 2c^{14}s(C_R) \leq c^{27}(\alpha(C_R) + s(C_R)) \leq c^{28}H$$

Finally, we prove the third part of the theorem. Consider some vertex v, belonging to some component C'. Let ψ be the embedding computed by the algorithm. Then $|\psi(v) - \psi(r)| \leq |\psi(v) - \psi(r(C'))| + |\psi(r) - \psi(r(C'))|$, while $D(v, r) = D(v, r(C')) + D(r, r(C'))$. By the inductive hypothesis, $|\psi(v) - \psi(r(C'))| \leq c^{30}D(v, r(C'))$. We now prove that $|\psi(r) - \psi(r(C'))| \leq c^{30}D(r, r(C'))$, thus finishing the proof.

The distance between the embeddings of $r(C')$ and r consists of three parts: (1) The length of the recursive embedding of component C' to the right of its root $r(C')$: bounded by $c^{28}D(r, r(C'))$ by the induction hypothesis; the empty space we need to leave between the embedding of C' and its neighbor that lies between C' and r: bounded by $6c^5\alpha(C')$; (3) the embeddings of all the components lying between C' and the root r, and their empty spaces. The last term can be bounded by the similar way we used to bound the distance between the embedding of C_R and the root, which is at most $c^{27}\alpha(C_R)$. Summing the three terms together, we get:

$$|\psi(r) - \psi(r(C'))| \leq c^{28}D(r, r(C')) + 6c^5\alpha(C) + c^{27}\alpha(C_R) \leq c^{29}D(r, r(C'))$$

□

Theorem 5. *(Theorem 2) The algorithm produces a non-contracting embedding with distortion bounded by* $c^{O(1)}$.

Proof: It is easy to see that the embedding produced by the algorithm is non-contracting. We now prove that the distortion is at most $4c^{32}$. Let $e = (u, v)$ be some edge in our original instance. Let C be the first instance in our recursive algorithm executions, where u and v are separated: i.e., $u, v \in C$, but there are two components $C_u, C_v \subseteq C$, such that $u \in C_u$, $v \in C_v$. Let r denote the root of the current instance.

Then edge e is a large or a medium-size edge, and thus $D(u, v) = w_e \geq \frac{d(u)}{c^2}$. Also, since $d(v) = d(u) + w_e \leq c^2 w_e + w_e \leq 2c^2 w_e$, we have that in total:

$$D(u, v) = w_e \geq \frac{d(u) + d(v)}{4c^2}$$

On the other hand, consider the embedding ψ produced by our algorithm. Then:

$$\begin{aligned} |\psi(u) - \psi(v)| &\leq |\psi(u) - \psi(r)| + |\psi(v) - \psi(r)| \\ &\leq c^{30}(d(u) + d(v)) \\ &\leq 4c^{32}\frac{d(u) + d(v)}{4c^2} \\ &\leq 4c^{32}w_e \end{aligned}$$

□