Homework Assignment 2

Due: Monday, May 16 in class.

Homework Policy: You are free to discuss the problems with other students and consult online material. However, you must write up your own solutions in your own words and mention the names of the people you discussed them with, and sources you consulted.

You can use any results that were proved in class.

1. Let G be the $(k \times k)$-grid and let U be the set of all vertices lying in the first row of G. Prove that U is node-well-linked in G.

2. Let G be an α-expander with maximum vertex degree d, where both α and d are constants. Prove that G contains a clique minor of size $\Omega(n^{1/3})$. (Hint: use the separator theorem for excluded-minor graphs).

3. In this question we consider the maximum-weight vertex cover problem in bipartite graphs: given a bipartite graph $G = (V_1, V_2, E)$ with weights $w_v \geq 0$ on its vertices, find a minimum-weight subset U of vertices, such that for every edge $e \in E$, at least one endpoint of e is in U. Consider the following LP-relaxation of the problem:

$$
\begin{array}{ll}
\min & \sum_{v \in V} w_v x_v \\
\text{s.t.} & \\
& \begin{array}{rl} x_v + x_u \geq 1 & \forall (u,v) \in E \\ x_v \geq 0 & \forall v \in V \end{array}
\end{array}
$$

   (a) Let x be any basic feasible solution to the LP. Let $E' \subseteq E$ be any set of edges, such that the constraints corresponding to the edges of E' are linearly independent. Prove that $G[E']$ cannot contain cycles.

   (b) Show that in any basic feasible solution x, there is a vertex v with $x_v = 0$.

   (c) Design an efficient LP-rounding algorithm that solves the problem exactly. Prove the algorithm's correctness.

4. Suppose we are given a graph G, a laminar family $\mathcal{L}$ of subsets of its vertices, and some additional set $B \subseteq V(G)$. Let $A \in \mathcal{L}$ be some set, such that A and B cross. Denote by N the number of sets in $\mathcal{L}$ that B crosses, by N_1 the number of sets in $\mathcal{L}$ that $A \cup B$ crosses, and by N_2 the number of sets in $\mathcal{L}$ that $A \cap B$ crosses. Prove that $N_1 < N$ and $N_2 < N$. Hint: show that if $X \in \mathcal{L}$ crosses $A \cup B$, then it also crosses B (and same for $A \cap B$).