Homework set 3

Note: the homework sets are not for submission. They are designed to help you prepare for the quizzes. It is **highly recommended** that you solve all problems and write the solutions down.

1. We are given a directed graph $G = (V, E)$, with two special vertices s and t, and arbitrary non-negative capacities $c(e)$ on edges $e \in E$. Additionally, we are given a valid flow f: that is, for each edge $e \in E$, we have a flow value $f(e)$, such that the edge capacity constraints and the flow conservation constraints are satisfied. Moreover, flow f is acyclic: that is, there is no cycle in G on which all edges carry positive flow. Our goal is to find a collection $\mathcal{P}$ of paths connecting s to t, together with values $f'(P) > 0$ for each path $P \in \mathcal{P}$, such that for each edge $e \in E$,

$$\sum_{\substack{P\in\mathcal{P}:\\ e\in P}} f'(P) = f(e).$$

   Show an efficient algorithm to find such collection of paths with the values $f'(P_i)$, prove its correctness, and analyze its running time.

   Remark: Such collection of paths is called a *flow-path decomposition* of the flow f.

2. In this question we study a variant of the Ford-Fulkerson algorithm. Recall that given a residual graph G_f and an s-t path P in G_f, we have denoted by $b_f(P) = \min_{e\in P}\{c_f(e)\}$ - the minimum residual capacity of any edge on P. We run the standard Ford-Fulkerson algorithm, except that we choose augmenting paths according to the following rule: select a path P with maximum value $b_f(P)$, breaking ties arbitrarily. For each iteration i of the algorithm, let b_i denote the value $b_f(P)$ of the path P selected in iteration i. Prove or disprove: The values b_i, for $i \geq 1$, always form a non-increasing sequence.

   Hint: the statement is false.

3. We are given a flow network $G = (V, E)$, with positive integral capacities $c(e)$ on edges $e \in E$, a source s and a sink t. Recall that an s-t cut in G is a partition (A, B) of the vertices of V, such that $s \in A$, $t \in B$. An s-t cut (A, B) is a minimum cut iff the value $C(A, B)$ is minimal among all s-t cuts. Notice that it is possible for a graph to contain several minimum cuts.

   - Show an example of a graph G, that contains $\Omega(n^2)$ minimum s-t cuts, where $n = |V|$.
   - Show an example of a graph G that contains a unique minimum s-t cut (that is, the number of minimum s-t cuts in G is 1).
   - Show an efficient algorithm to determine whether G contains a unique minimum s-t cut, or the number of minimum cuts is greater than 1. Prove the algorithm's correctness.

4. In this question we will help a hospital figure out whether it has enough supplies for blood transfusions for its patients. There are x_A patients with blood type A, x_B patients with blood type B, x_{AB} patients with blood type AB, and x_O patients with blood type O currently at the hospital, and each patient needs a transfusion of one unit of blood. The hospital has at its disposal s_A units of blood of type A, s_B of type B, s_{AB} of type AB and s_O of type O. The rules of blood transfusion are as follows:

- Patients with blood type A can receive only blood of types A or O.
- Patients with blood type B can receive only blood types B or O.
- Patients with blood type O can receive only blood of type O.
- Patients with blood type AB can receive any of the four types.

Design an efficient algorithm that determines whether the hospital's blood supply is sufficient for treating the patients, and if so, computes a way to distribute the hospital supplies among the patients, so each of them receives blood of an appropriate type.