Algorithms Tutorial 1

January 4, 2011

Question 0.1. *Given a set of n jobs, where job i has processing time $p_i > 0$ and start time s_i, and only one job can run on one machine at a given time. Determine the minimum number of machines required to schedule all the jobs and also the schedule (i.e. the map from jobs to machines).*

This is the example on pg. 122 of the Kleinberg & Tardos book.

Question 0.2. *Given a set of n jobs, where job i has processing time $p_i > 0$ and weight $w_i > 0$, design an algorithm which outputs a schedule (i.e. a map from jobs to start times) which minimizes $\sum_{i=1}^{n} w_i C_i$ (C_i is the actual completion time for job i in your schedule). Note that here you have only a single machine.*

This is problem 13 on pg. 194 of the Kleinberg & Tardos book. As a hint try proving correctness/optimality using an exchange argument similar to the example on pg. 128.

Here is a sketch of the solution for qn 0.2.
Algorithm: Schedule the jobs in decreasing order of $\frac{w_i}{p_i}$ (ties broken arbitrarily).

Proof of optimality. Let R denote the schedule output by the algorithm above. W.l.o.g. we can assume that $R_k = k$ i.e. the kth job scheduled by R is labeled k. Suppose that O is some optimal schedule which differs from R for the first time at position k i.e. say $O_k = j$ and $O_i = R_i$ for $i < k$. Construct a new schedule P^k such that $P_i^k := O_i (= R_i)$ for $i < k$, $P_k^k := k$, and $P_i^k := O_{i-1}$ for $i > k$ and $O_{i-1} \neq k$ (i.e. make sure not to schedule job k twice).

Lemma 0.3.
$$\sum_{i=1}^{n} w_i C_i^{P^k} \leq \sum_{i=1}^{n} w_i C_i^O.$$

Proof. Note that translating the completion times $C_i^{P^k}$ and C_i^O by a constant amount does not change the direction of the inequality. So let the $t = 0$ be the completion time of job $k - 1$ (in both schedules). Let $O_m = k$ and $T := \sum_{i=k+1}^{m} p_{O_i}$. Observe that

$$\sum_i w_i C_i^{P^k} - \sum_i w_i C_i^O = -w_k(T + p_j) + p_k \sum_{i=k}^{m} w_{O_i}.$$

Since $\frac{w_k}{p_k} \geq \frac{w_{O_i}}{p_{O_i}}$ for $i \geq k$ we have $w_k(T + p_j) \geq p_k \sum_{i=k}^{m} w_{O_i}$. Hence the lemma follows. □

Observe that by construction $P_i^k = R_i$ for $i \leq k$. Repeated application of the lemma above allows us to construct optimal schedules P^k for all $k \leq n$, thus the schedule R costs no more than some optimal schedule O. Therefore the optimality of R follows. □