Mathematical Toolkit **Fall 2026**

Homework 2 Solutions

1. **Projections**.

   A linear operator $\varphi : V \to V$ is called a projection if $\varphi^2 = \varphi$ i.e., $\varphi^2(v) = \varphi(v)\ \forall v \in V$. For the parts below, let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $\varphi : V \to V$ be a projection.

   (a) Show that $\psi : V \to V$ defined as $\psi(v) = v - \varphi(v)$ is also a projection.

   (b) Show that $\ker(\varphi) = \text{im}(\psi)$ and $\text{im}(\varphi) = \ker(\psi)$.

   (c) Show that any $v \in V$ can be uniquely decomposed as $v = u + w$, with $u \in \text{im}(\varphi)$ and $w \in \ker(\varphi)$. We say that those two subspaces are complementary.

   (d) What are the possible eigenvalues λ of φ and the respective eigenspaces, i.e., $U_\lambda := \{v \mid \varphi(v) = \lambda v\}$?

   For the remaining parts, we will call a projection φ an orthogonal projection if $\ker(\varphi) = \text{im}(\varphi)^\perp$, where $W^\perp := \{v \in V \mid \langle v, w\rangle = 0\ \forall w \in W\}$ i.e., these two subspaces are orthogonal complements.

   (e) Let $V = \mathbb{R}^2$ with the usual inner product, show that $\varphi_1(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ and $\varphi_2(x, y) = \left(\frac{2x+y}{3}, \frac{2x+y}{3}\right)$ are projections with the same image. Are their kernels the same?

   (f) Which of φ_1 and φ_2 is orthogonal and which isn't? Can you suggest a different inner product on $\mathbb{R}^2$ which flips your answer?

   Solution:

   (a) **Proof:** $\psi(\psi(v)) = \psi(v) - \varphi(\psi(v)) = v - \varphi(v) - \varphi(v) + \varphi(\varphi(v)) = v - \varphi(v) = \psi(v)$. ■

   (b) **Proof:** $\forall v \in ker(\varphi)$, $\varphi(v) = 0$, therefore, $v = v - \varphi(v) \in im(\psi)$.
   $\forall v \in im(\psi)$, $\exists w \in V$ s.t. $v = \psi(w) = w - \varphi(w)$. Then $\varphi(v) = \varphi(w) - \varphi(\varphi(w)) = 0$, therefore $v \in ker(\varphi)$.
   Hence $ker(\varphi) = im(\psi)$.
   $\forall v \in im(\varphi)$, $\exists w \in V$ s.t. $v = \varphi(w)$. Then $\psi(v) = \varphi(w) - \varphi(\varphi(w)) = 0$. Hence, $v \in ker(\psi)$.

$\forall v \in ker(\psi)$, $\psi(v) = v - \varphi(v) = 0$, therefore, $v = \varphi(v) \in im(\varphi)$.
Hence, $im(\varphi) = ker(\psi)$. ■

(c) We can decompose v as $v = \varphi(v) + v - \varphi(v)$. $\varphi(v) \in im(\varphi)$ and $v - \varphi(v) \in im(\psi(v)) = ker(\varphi)$. If this is not unique, then $\exists y \in V$, $u = \varphi(y) \neq \varphi(v)$ and $w := v - \varphi(y) \in ker(\varphi)$. Therefore $\varphi(w) = \varphi(v) - \varphi(y) = 0$, which contradicts that $\varphi(y) \neq \varphi(v)$.

(d) $\lambda = 1$ and $im(\varphi)$ is its respective eigenspace. $\lambda = 0$ and $ker(\varphi)$ is its respective eigenspace.

(e) Both images are $V_{im} = \{(x, x) | x \in \mathbb{R}\}$.

Proof: It is easy to check that $im(\varphi_1) \subseteq V_{im}$ and $im(\varphi_2) \subseteq V_{im}$ since $\frac{x+y}{2} = \frac{x+y}{2}$ and $\frac{2x+y}{3} = \frac{2x+y}{3}$.
For any $(x, x) \in V_{im}$, we can find $\varphi_1(2x, 0) = (x, x)$ and $\varphi_2(3x/2, 0) = (x, x)$. ■

Their kernels are different. $ker(\varphi_1) = \{(x, -x) | x \in \mathbb{R}\}$ while $ker(\varphi_2) = \{(x, -2x) | x \in \mathbb{R}\}$.

(f) φ_1 is while φ_2 not. We can define a different inner product as $\langle (x, y), (u, v) \rangle = 2xu + yv$.

2. **Adjoints**.

Let V, W be finite-dimensional inner product spaces $\varphi : V \to W$ and let $\varphi^* : W \to V$ denote the adjoint of φ. Prove that $(\varphi^*)^* = \varphi$.

Solution:

Proof: $\forall v \in V$, $\forall w \in W$, $\langle w, \varphi(v) \rangle = \langle \varphi^*(w), v \rangle$ and $\langle v, \varphi^*(w) \rangle = \langle \varphi^{**}(v), w \rangle$. $\langle w, \varphi(v) \rangle = \langle w, \varphi^{**}(v) \rangle$. Let $w = \varphi(v) - \varphi^{**}(v)$, we have $\langle \varphi(v) - \varphi^{**}(v), \varphi(v) - \varphi^{**}(v) \rangle = 0$ and thus, $\varphi(v) = \varphi^{**}(v)$, i.e. $\varphi = \varphi^{**}$. ■

3. **Various**.

The following are some useful results we have sketched in class and also used some of them in proofs. Here, you will prove them. For all the parts below, let V be a finite dimensional inner-product space with dimension n.

(a) Let $\{w_1, \ldots, w_k\}$ be an orthonormal subset of V. Show that it can be completed to an orthonormal basis of V i.e., there exist vectors $\{w_{k+1}, \ldots, w_n\}$ such that the set $\{w_1, \ldots, w_n\}$ forms an orthonormal basis of V.

(b) Let $\{w_1, \ldots, w_n\}$ and $\{u_1, \ldots, u_n\}$ be two different orthonormal bases of V. Prove that for all $v_1, v_2 \in V$

$$\sum_{i=1}^{n} \overline{\langle w_i, v_1 \rangle} \cdot \langle w_i, v_2 \rangle \;=\; \sum_{i=1}^{n} \overline{\langle u_i, v_1 \rangle} \cdot \langle u_i, v_2 \rangle \;.$$

Note that this implies that for all $v \in V$, $\sum_{i=1}^{n} |\langle w_i, v\rangle|^2 = \sum_{i=1}^{n} |\langle u_i, v\rangle|^2$.

(c) We call $\varphi : V \to V$ a unitary operator if $\varphi\varphi^* = \varphi^*\varphi = \mathrm{id}$. Show that φ is a unitary operator *if and only if* for any orthonormal basis $\{w_1, \ldots, w_n\}$ of V, the set $\{\varphi(w_1), \ldots, \varphi(w_n)\}$ is also an orthonormal basis. Hint: use the Real Spectral Theorem.

Solution:

(a) **Proof:** According to Proposition 1.3 in Lecture 3, we know that there exists a basis B of V containing $\{w_1, \cdots, w_k\}$. Let $B = \{w_1, \cdots, w_k, v_{k+1}, \cdots, v_n\}$. Similar to the proof of Gram-Schmidt orthogonalization, for $i = k+1, \cdots, n$, define

$$u_i = v_i - \sum_{j=1}^{i-1} \langle w_j, v_i\rangle \cdot w_j,$$
$$w_i = \frac{u_i}{\|u_i\|}.$$

It is easy to check that $\langle w_i, w_j\rangle = 0$ for all $j < i$. Therefore, $\{w_1, \cdots, w_n\}$ are mutually orthogonal and forms an orthonormal basis of V. ■

(b) **Proof:** We can represent $v_1 = \sum_{i=1}^{n} \langle w_i, v_1\rangle w_i$ and $v_2 = \sum_{i=1}^{n} \langle w_i, v_2\rangle w_i$. Then we have $\overline{\langle v_1, v_2\rangle} = \langle \sum_{i=1}^{n} \langle w_i, v_1\rangle w_i, \sum_{i=1}^{n} \langle w_i, v_2\rangle w_i\rangle = \sum_{i=1}^{n} \langle \langle w_i, v_1\rangle w_i, \langle w_i, v_2\rangle w_i\rangle = \sum_{i=1}^{n} \overline{\langle w_i, v_1\rangle} \cdot \langle w_i, v_2\rangle$.

Similarly, $\langle v_1, v_2\rangle = \sum_{i=1}^{n} \overline{\langle u_i, v_1\rangle} \cdot \langle u_i, v_2\rangle$. Therefore, $\sum_{i=1}^{n} \overline{\langle w_i, v_1\rangle} \cdot \langle w_i, v_2\rangle = \sum_{i=1}^{n} \overline{\langle u_i, v_1\rangle} \cdot \langle u_i, v_2\rangle$ ■

(c) **Proof:** "$\Rightarrow$"

For any orthonormal basis $\{w_1, \cdots, w_n\}$ of V, $\langle \varphi(w_i), \varphi(w_j)\rangle = \langle \varphi^*\varphi(w_i), w_j\rangle = \langle w_i, w_j\rangle$. $\langle \varphi(w_i), \varphi(w_j)\rangle = 0$ for $i \neq j$ and $\langle \varphi(w_i), \varphi(w_j)\rangle = 1$ for $i = j$. We have $\{\varphi(w_1), \cdots, \varphi(w_n)\}$ is orthonormal and $dim(Span(\{\varphi(w_1), \cdots, \varphi(w_n)\})) = n$. Thus $\{\varphi(w_1), \cdots, \varphi(w_n)\}$ is an orthonormal basis of V.

"$\Leftarrow$"

Let $\{w_1, \cdots, w_n\}$ be an eigenvector basis of $\varphi^*\varphi$ corresponding to eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ as $\varphi^*\varphi$ is self-adjoint. $\lambda_i = \langle w_i, \varphi^*\varphi(w_i)\rangle = \langle \varphi(w_i), \varphi(w_i)\rangle = 1$ for all $i = 1, \cdots, n$. For any $v = \sum_{i=1}^{n} c_i w_i \in V$, $\varphi^*\varphi(v) = \varphi^*\varphi(\sum_{i=1}^{n} c_i w_i) = \sum_{i=1}^{n} c_i \lambda_i w_i = v$. Therefore, $\varphi^*\varphi = id$.

Similarly, $\varphi\varphi^* = id$. ■

4. **Eigenvalue interlacing**.

Let α be a self-adjoint operator on an n-dimensional inner-product space V, and let

$w_0 \in V \setminus 0$ be a non-zero vector with $\|w_0\| = 1$. Let $W \subseteq V$ denote the subspace defined as $W := \{v \in V \mid \langle w_0, v\rangle = 0\}$. Let $\beta : W \to V$ be defined as

$$\beta(w) \;=\; \alpha(w) - \langle w_0, \alpha(w)\rangle \cdot w_0\,.$$

(a) Show that β is in fact an operator from W to W i.e., for all $w \in W$, we have $\beta(w) \in W$.

(b) Show that $\beta : W \to W$ as defined above is self-adjoint.

(c) Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of α and let $\mu_1 \geq \cdots \geq \cdots \mu_{n-1}$ denote the eigenvalues of β (since $\dim(W) = n-1$). Show that $\lambda_1 \geq \mu_1$.

(d) Show that the eigenvalues of α and β are *interlacing* i.e.,

$$\lambda_i \;\geq\; \mu_i \;\geq\; \lambda_{i+1} \quad \forall i \in [n-1]\,.$$

Solution:

(a) **Proof:** $\forall w \in W$, $\langle w_0, \beta(w)\rangle = \langle w_0, \alpha(w) - \langle w_0, \alpha(w)\rangle \cdot w_0\rangle = \langle w_0, \alpha(w)\rangle - \|w_0\|^2 \langle w_0, \alpha(w)\rangle = 0$. Therefore, $\beta(w) \in W$. ■

(b) **Proof:** $\forall w, u \in W$, $\langle w, \beta(u)\rangle = \langle w, \alpha(u) - \langle w_0, \alpha(u)\rangle \cdot w_0\rangle = \langle w, \alpha(u)\rangle - \langle w, \langle w_0, \alpha(u)\rangle \cdot w_0\rangle = \langle \alpha(w), u\rangle = \langle \alpha(w) - \langle w_0, \alpha(w)\rangle \cdot w_0, u\rangle = \langle \beta(w), u\rangle$. ■

(c) **Proof:** $\lambda_1 = \max_{v\in V} \frac{\langle v, \alpha(v)\rangle}{\|v\|^2}$ and $\mu_1 = \max_{v\in W} \frac{\langle v, \beta(v)\rangle}{\|v\|^2} = \max_{v\in W} \frac{\langle v, \alpha(v)\rangle}{\|v\|^2}$. As $W \subseteq V$, $\lambda_1 \geq \mu_1$. ■

(d) **Proof:** We prove it by induction. Let $\{v_1, \cdots, v_n\}$ denote an orthonormal eigenvector basis of α for V corresponding to $\lambda_1, \cdots, \lambda_n$ and $\{w_1, \cdots, w_{n-1}\}$ denote an orthonormal eigenvector basis of α for W corresponding to $\mu_1, \cdots, \mu_{n-1}$.

Base case: from solution of (c), we know that $\lambda_1 \geq \mu_1$. Also, $dim(Span(\{v_1, v_2\}) \cap W) \geq dim(Span(\{v_1, v_2\})) + dim(W) - n = 1$. Thus, there exists a $w = c_1 v_1 + c_2 v_2 \in Span(\{v_1, v_2\}) \cap W \backslash \{0\}$. Then,

$$\mu_1 \geq \frac{\langle w, \beta(w)\rangle}{\|w\|^2} = \frac{\langle w, \alpha(w)\rangle}{\|w\|^2} = \frac{c_1^2\lambda_1 + c_2^2\lambda_2}{c_1^2 + c_2^2} \geq \lambda_2.$$

Induction: if for any $i \leq k$, we have $\lambda_{i-1} \geq \mu_{i-1} \geq \lambda_i$. Then we want to prove that $\lambda_k \geq \mu_k \geq \lambda_{k+1}$. $dim(Span(\{v_1, \cdots, v_{k+1}\}) \cap Span(\{w_k, \cdots, w_{n-1}\})) \geq dim(Span(v_1, \cdots, v_{k+1})) + dim(Span(w_k, \cdots, w_{n-1})) - n = 1$. Thus, there exists a $w = \sum_{i=1}^{k+1} c_i v_i \in Span(\{v_1, \cdots, v_{k+1}\}) \cap Span(\{w_k, \cdots, w_{n-1}\}) \backslash \{0\}$. Then,

$$\mu_k \geq \frac{\langle w, \beta(w)\rangle}{\|w\|^2} = \frac{\langle w, \alpha(w)\rangle}{\|w\|^2} = \frac{\sum_{i=1}^{k+1} c_i^2\lambda_i}{\sum_{i=1}^{k+1} c_i^2} \geq \lambda_{k+1}.$$

Similarly, $dim(Span(\{v_k, \cdots, v_n\}) \cap Span(\{w_1, \cdots, w_k\})) \geq dim(Span(\{v_k, \cdots, v_n\})) + dim(Span(\{w_1, \cdots, w_k\})) - n = 1$ and thus, $\lambda_k \geq \mu_k$. ■