Mathematical Toolkit	**Spring 2023**
Homework 5	
Due: May 17, 2023	

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX. Homeworks are due by the start of class on the due date.*

1. **Randomization and Non-Uniformity.**

   Prove that **BPP** $\subseteq$ **P/poly**. Hint: Use Chernoff-Hoeffding bounds.

2. **Gaussian Random Variables.**

   Prove the following useful facts about Gaussian random variables:

   (a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ be two vectors. Let $\mathbf{g} \in \mathbb{R}^n$ be a random vector such that each coordinate g_i of $\mathbf{g}$ is distributed as a Gaussian random variable with mean 0 and variance 1, and any two coordinates g_i, g_j (for $i \neq j$) are independent. Then show that

   $$\mathop{\mathbb{E}}_{\mathbf{g}} \left[\langle \mathbf{u}, \mathbf{g} \rangle \cdot \langle \mathbf{v}, \mathbf{g} \rangle \right] \;=\; \langle \mathbf{u}, \mathbf{v} \rangle \,.$$

   (b) Let g be a Gaussian random variable with mean 0 and variance 1. Show that for any $t \in \mathbb{R}$, we have

   $$\mathbb{E}\left[e^{tg}\right] \;=\; e^{t^2/2} \,.$$

3. **Supremum of Gaussians.**

   (a) Let $g \sim N(0,1)$ be a Gaussian random variable with mean 0 and variance 1. Show that for $t > 0$,

   $$\mathbb{P}\left[g \geq t\right] \;\leq\; e^{-t^2/2} \,.$$

   Hint: Use 2(b).

   (b) Let $g_1, \ldots, g_n \sim N(0,1)$ be independent Gaussian random variables. Show that for some constants c_1, c_2 we have

   $$\mathbb{E}\left[\max_{i \in [n]} |g_i|\right] \;\leq\; c_1 \sqrt{\ln n} + c_2 \,.$$

   You may use the fact that for a non-negative random variable Z, the expectation can be computed as $\mathbb{E}\left[Z\right] = \int_0^\infty \mathbb{P}\left[Z \geq t\right] dt$.