Mathematical Toolkit	**Spring 2023**

Homework 2

Due: April 10, 2023

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX. Homeworks are due by the start of class on the due date.*

1. **Projections**.

   A linear operator $\varphi : V \to V$ is called a projection if $\varphi^2 = \varphi$ i.e., $\varphi^2(v) = \varphi(v)\ \forall v \in V$. For the parts below, let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $\varphi : V \to V$ be a projection.

   (a) Show that $\psi : V \to V$ defined as $\psi(v) = v - \varphi(v)$ is also a projection.

   (b) Show that $\ker(\varphi) = \mathrm{im}(\psi)$ and $\mathrm{im}(\varphi) = \ker(\psi)$.

   (c) Show that any $v \in V$ can be uniquely decomposed as $v = u + w$, with $u \in \mathrm{im}(\varphi)$ and $w \in \ker(\varphi)$. We say that those two subspaces are complementary.

   (d) What are the possible eigenvalues λ of φ and the respective eigenspaces, i.e., $U_\lambda := \{v \mid \varphi(v) = \lambda v\}$?

   For the remaining parts, we will call a projection φ an orthogonal projection if $\ker(\varphi) = \mathrm{im}(\varphi)^\perp$, where $W^\perp := \{v \in V \mid \langle v, w \rangle = 0\ \forall w \in W\}$ i.e., these two subspaces are orthogonal complements.

   (e) Let $V = \mathbb{R}^2$ with the usual inner product, show that $\varphi_1(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2}\right)$ and $\varphi_2(x, y) = \left(\frac{2x+y}{3}, \frac{2x+y}{3}\right)$ are projections with the same image. Are their kernels the same?

   (f) Which of φ_1 and φ_2 is orthogonal and which isn't? Can you suggest a different inner product on $\mathbb{R}^2$ which flips your answer?

2. **Adjoints**.

   Let V, W be finite-dimensional inner product spaces $\varphi : V \to W$ and let $\varphi^* : W \to V$ denote the adjoint of φ. Prove that $(\varphi^*)^* = \varphi$.

3. **Various**.

   The following are some useful results we have sketched in class and also used some of them in proofs. Here, you will prove them. For all the parts below, let V be a finite dimensional inner-product space with dimension n.

   (a) Let $\{w_1, \ldots, w_k\}$ be an orthonormal subset of V. Show that it can be completed to an orthonormal basis of V i.e., there exist vectors $\{w_{k+1}, \ldots, w_n\}$ such that the set $\{w_1, \ldots, w_n\}$ forms an orthonormal basis of V.

   (b) Let $\{w_1, \ldots, w_n\}$ and $\{u_1, \ldots, u_n\}$ be two different orthonormal bases of V. Prove that for all $v_1, v_2 \in V$

   $$\sum_{i=1}^{n} \overline{\langle w_i, v_1 \rangle} \cdot \langle w_i, v_2 \rangle \;=\; \sum_{i=1}^{n} \overline{\langle u_i, v_1 \rangle} \cdot \langle u_i, v_2 \rangle \;.$$

   Note that this implies that for all $v \in V$, $\sum_{i=1}^{n} |\langle w_i, v \rangle|^2 \;=\; \sum_{i=1}^{n} |\langle u_i, v \rangle|^2$.

   (c) We call $\varphi : V \to V$ a unitary operator if $\varphi\varphi^* = \varphi^*\varphi = \mathrm{id}$. Show that φ is a unitary operator *if and only if* for any orthonormal basis $\{w_1, \ldots, w_n\}$ of V, the set $\{\varphi(w_1), \ldots, \varphi(w_n)\}$ is also an orthonormal basis. Hint: use the Real Spectral Theorem.

4. **Eigenvalue interlacing**.

   Let α be a self-adjoint operator on an n-dimensional inner-product space V, and let $w_0 \in V \setminus 0$ be a non-zero vector with $\|w_0\| = 1$. Let $W \subseteq V$ denote the subspace defined as $W := \{v \in V \mid \langle w_0, v \rangle = 0\}$. Let $\beta : W \to V$ be defined as

   $$\beta(w) \;=\; \alpha(w) - \langle w_0, \alpha(w) \rangle \cdot w_0 \,.$$

   (a) Show that β is in fact an operator from W to W i.e., for all $w \in W$, we have $\beta(w) \in W$.

   (b) Show that $\beta : W \to W$ as defined above is self-adjoint.

   (c) Let $\lambda_1 \geq \cdots \geq \lambda_n$ denote the eigenvalues of α and let $\mu_1 \geq \cdots \geq \cdots \mu_{n-1}$ denote the eigenvalues of β (since $\dim(W) = n - 1$). Show that $\lambda_1 \geq \mu_1$.

   (d) Show that the eigenvalues of α and β are *interlacing* i.e.,

   $$\lambda_i \;\geq\; \mu_i \;\geq\; \lambda_{i+1} \quad \forall i \in [n-1] \,.$$