Mathematical Toolkit	**Spring 2023**

Homework 1

Due: March 29, 2023

Note: *You may discuss these problems in groups. However, you must write up your own solutions and mention the names of the people in your group. Also, please do mention any books, papers or other sources you refer to. It is recommended that you typeset your solutions in LaTeX. Homeworks are due by the start of class on the due date.*

1. **Field trip**.

   Recall that for a prime p, $\mathbb{F} = \mathbb{Q}^2$ (pairs of rational numbers) is a field with the notions of addition and multiplication defined as

   $$(a,b) + (c,d) = (a+c, b+d) \quad \text{and} \quad (a,b) \cdot (c,d) = (ac + pbd, ad + bc)\,.$$

   (a) What are the additive and multiplicative identities? What is the multiplicative inverse of (a,b) for $(a,b) \neq 0_{\mathbb{F}}$?

   (b) Does everything still go through for $p = 6$? How about $p = 4$?

   (c) When p is such that $\mathbb{F}$ defined above is a field, the set

   $$S = \{a + b\sqrt{p} \mid a, b \in \mathbb{Q}\}$$

   can be thought of as a vector space over the field $\mathbb{Q}$. What is its dimension? What is a basis for it?

2. **Basis Basics**.

   If $S = \{v_1, ..., v_n\}$ is a basis for V, then we know that any $v \in V$ is in the span of S and so can be written as $a_1 v_1 + ... + a_n v_n$ for some $a_1, ..., a_n$.

   (a) Prove this decomposition is unique.

   (b) Give an example of a case of a non-unique decomposition (i.e., multiple ways of writing v as a linear combination of vectors in S) when S is *not* a linearly independent set.

3. **Linear equations**.

   Let $A \in \mathbb{F}_2^{m \times n}$ be a matrix with entries in the field $\mathbb{F}_2$ and let $m < n$ (m rows and n columns). Let all rows of A be linearly independent in the vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2$.

(a) What is the dimension of the space $\ker(A)$?

(b) How many vectors $x \in \mathbb{F}_2^n$ satisfy the system of equations $Ax = 0$? (Note that here 0 denotes the zero vector in $\mathbb{F}_2^m$.)

(c) Let $b \in \mathbb{F}_2^m$ be such that the system of equations $Ax = b$ has at least one solution, say x_0. Show that $\{x - x_0 \mid Ax = b\} = \ker(A)$. What is the total number of solutions to the system $Ax = b$?

For this problem you may use the fact that for a matrix $A \in \mathbb{F}^{m\times n}$ for any field $\mathbb{F}$, if $R \subseteq \mathbb{F}^n$ denotes the set of its rows and $C \subseteq \mathbb{F}^m$ denotes the set of its columns, then

$$\dim(\mathrm{Span}\,(R)) \;=\; \dim(\mathrm{Span}\,(C))\,.$$

The quantity $\dim(\mathrm{Span}\,(R))$ is called the row-rank of A and $\dim(\mathrm{Span}\,(C))$ is called the column-rank of A.

4. **Inner Products**.

Consider the vector space $\mathbb{R}[x]$ of polynomials in a single variable x with coefficients in $\mathbb{R}$. Define the function $\mu : \mathbb{R}[x] \times \mathbb{R}[x] \to \mathbb{R}$ as

$$\mu(P, Q) \;=\; \mathrm{degree}(P \cdot Q) \quad \text{for all } P, Q \in \mathbb{R}[x]\,,$$

where $P \cdot Q$ denotes the product of the two polynomials P and Q (which is another polynomial). Is the function μ an inner product? Justify your answer.

5. **Eigenvalues**.

Let V be a finite dimensional vector space over a field $\mathbb{F}$ and $\alpha, \beta : V \to V$ be linear operators. Show that for every $\lambda \in \mathbb{F}$ (including $0_{\mathbb{F}}$), λ is an eigenvalue of $\alpha\beta$ if and only if λ is an eigenvalue of $\beta\alpha$. Here, $\alpha\beta$ denotes the linear transformation $\alpha \circ \beta$ defined as $\alpha\beta(v) = \alpha(\beta(v))\ \forall v \in V$ (and $\beta\alpha$ is defined similarly).