Lecture notes

An Integer program (IP) is an optimization problem of the form

$$\max \; c^T x \quad \text{s.t.} \quad \begin{array}{l} Ax \le b \\ x \geq 0 \\ x \in \mathbb{Z}^n \end{array} \; ; \quad \begin{array}{l} A \in \mathbb{Z}^{m \times n} \\ b \in \mathbb{Z}^m \\ c \in \mathbb{R}^n \end{array}$$

LP relaxation: $\max c^T x$ s.t. $Ax \le b$, $x \geq 0$, $x \in \mathbb{R}^n$.

<u>Fact.</u> Let x^*_{IP} be the IP opt. soln., x^*_{LP} the LP optimum.

Then, $c^T x^*_{LP} \geq c^T x^*_{IP}$.

<u>Fact.</u> LPs can be solved efficiently in theory and in practice. (simplex, interior point methods, ...)

<u>Fact.</u> Let $\hat{x}$ be the opt. soln. to LP $\max c^T x$ s.t. $\begin{array}{l} Ax \le b \\ x \geq 0 \\ \hat{a}^T x \le \hat{b} \\ x \in \mathbb{R}^n \end{array}$.

Then, $c^T x^*_{LP} \geq c^T \hat{x}$.

(further constraining the LP can only decrease objective).

Branch-and-Bound for solving IPs.

(assumed to have a finite optimum).

$$N_0 = \boxed{LP_0 = \begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \\ & x \ge 0 \\ & x \in \mathbb{R}^n \end{array}}$$

Algorithm. [Vanilla B & B]

Input: IP ~~initial feasible soln x^*~~

Initialize: ~~z^*~~, $\mathcal{L} = \{N_0\}$, $z^* = -\infty$, $x^* = 0$

While $\mathcal{L} \neq \emptyset$:

- Choose node N_i from $\mathcal{L}$ & delete it from $\mathcal{L}$.
- Solve LP_i.
- If LP_i is infeasible:
  - ~~continue~~ "Prune" N_i.
- Else:
  - Let x^i be opt. soln to LP_i, $z_i \leftarrow c^T x^i$.
  - If $z_i \le z^*$:
    - Prune N_i.
  - Else if: $z_i > z^*$ AND $x^i \in \mathbb{Z}^n$
    - $x^* \leftarrow x^i$, $z^* \leftarrow z_i$
    - Prune N_i
  - Else if: $z_i > z^*$ AND $x^i \notin \mathbb{Z}^n$
    - Choose j s.t. $x^i_j \notin \mathbb{Z}$
    - $N_{i1} \leftarrow N_i$ w/ additional constr. $x_j \le \lfloor x^i_j \rfloor$
    - $N_{i2} \leftarrow N_i$ " " " $x_j \ge \lceil x^i_j \rceil$
    - $\mathcal{L} \leftarrow \mathcal{L} \cup \{N_{i1}, N_{i2}\}$

Return x^*.

B & B example.

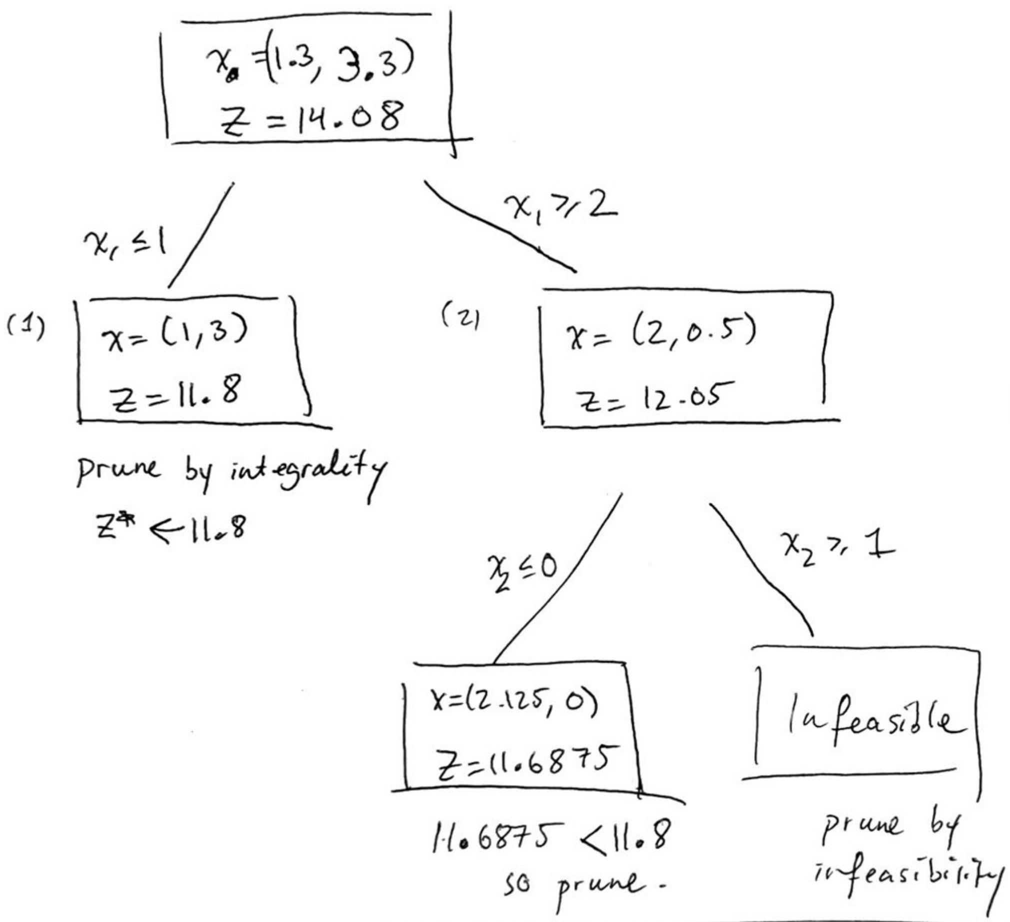
$$\max \; 5.5x_1 + 2.1x_2$$

$$\text{s.t.} \quad -x_1 + x_2 \le 2$$

$$8x_1 + 2x_2 \le 17$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \in \mathbb{Z}^2.$$



Branch on x_1.

$12.05 > 11.8$, so expand (2) further

Branch on x_2

So, $x^* = (1,3)$

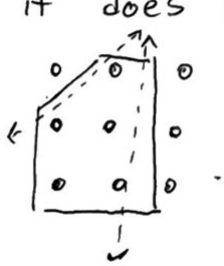
$z^* = 11.8$

Cuts

Let $P = \{x \in \mathbb{R}^n : Ax \le b, x \geq 0\}$, $P_I = P \cap \mathbb{Z}^n$.

Defn. IP $\max\{c^T x : x \in P_I\}$. A <u>valid cutting plane</u> is a constraint $\alpha^T x \le \beta$ s.t. $\alpha^T x \le \beta \;\; \forall x \in P_I$.

I.e., it does not chop off any integer feasible points.



In B & B, if $\alpha^T x^i > \beta$, can get a better LP bound.

Chvátal – Gomory cut. Parametrized by $u \in [0,1]^m$, is the cut s.t.

$$\lfloor u^T A \rfloor x \le \lfloor u^T b \rfloor .$$

So if $A = \begin{bmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{bmatrix}$, this is $\sum_{i=1}^{n} \lfloor u^T a_i \rfloor x_i \le \lfloor u^T b \rfloor$.

Fact. CG cuts are valid.

Proof: $Ax \le b \;\; \forall x \in P \Rightarrow uAx \le u^T b \;\; \forall x \in P$

$$\Rightarrow \lfloor uA \rfloor x \le u^T b \;\; \forall x \in P.$$

$$\Rightarrow \lfloor uA \rfloor x \le \lfloor u^T b \rfloor \;\; \forall x \in P \cap \mathbb{Z}^n. \qquad \square$$

Pseudodim. of B&B trees w/ a single CG cut at the root.

Let $f_u(c, A, b)$ = size of B&B tree after applying single CG cut $\lfloor u^T A \rfloor x \le \lfloor u^T b \rfloor$ at root.

Want to bound $\mathrm{Pdim}(\{ f_u : u \in [0,1]^m \})$.

Dual fn. $f^*_{c,A,b}(u) = f_u(c, A, b)$.

Thm. [Sensitivity] For any $n \in \mathbb{N}$, $\exists$ an IP (c, A, b) with two constraints and n variables s.t. if $\frac{1}{2} \le u_1 - u_2 < \frac{n+1}{2n}$, $f_u(c,A,b) = 1$, if $\frac{n+1}{2n} \le u_1 - u_2 < 1$, $f_u(c,A,b) \ge 2^{(n-1)/2}$

Lemma [Piecewise-constant duals].

For any (c, A, b), there are $O(\|A\|_{1,1} + \|b\|_1 + n)$ hyperplanes that partition $[0,1]^m$ into regions s.t. for any region R, $f^*_{c,A,b}(u)$ is constant $\forall\, u \in R$. $\left(\|A\|_{1,1} = \sum_{i,j} |A[i,j]|,\ \|b\|_1 = \sum_i |b_i|\right)$.

Proof: Let $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$. For any $u \in [0,1]^m$,

$$\lfloor u^T a_i \rfloor \in \left[-\|a_i\|_1, \|a_i\|_1\right]$$

$$\lfloor u^T b \rfloor \in \left[-\|b\|_1, \|b\|_1\right].$$

For ~~region~~ each integer $k_i \in \left[-\|a_i\|_1, \|a_i\|_1\right]$,

$$\lfloor u^T a_i \rfloor = k_i \iff k_i \le u^T a_i < k_i + 1.$$

Collect these halfspaces $\forall\, i$. (total of $\sum_{i=1}^n 2\|a_i\|_1 + 1 = O(\|A\|_{1,1} + n)$).

Similarly for each integer $l \in \left[-\|b\|_1, \|b\|_1\right]$,

$$\lfloor u^T b \rfloor = l \iff l \le u^T b < l + 1;$$ additional $2\|b\|_1 + 1$ halfspaces.

In any region: the vector $\left(\lfloor u^T a_1 \rfloor, \ldots, \lfloor u^T a_n \rfloor, \lfloor u^T b \rfloor\right)$ is invariant, and so is the resulting cut, and so is the resulting B&B tree.

Thm. Let $\mathcal{F}_{\alpha,\beta}$ denote the collection of all fns. f_u for $u \in [0,1]^m$ on IPs (c, A, b) with $\|A\|_{1,1} \le \alpha$, $\|b\|_1 \le \beta$. $\operatorname{Pdim}(\mathcal{F}_{\alpha,\beta}) = O\left(m \log\left(m(\alpha + \beta + n)\right)\right)$.