TTIC 31250
An Introduction to the Theory of Machine Learning

Learning from noisy data, intro to SQ model

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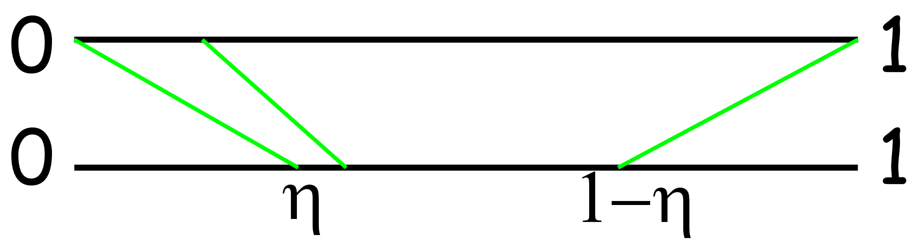
Learning when there is no perfect predictor

- Hoeffding/Chernoff bounds: minimizing training error will approximately minimize true error: just need $O(1/\varepsilon^2)$ samples versus $O(1/\varepsilon)$.
- What about polynomial-time algorithms? Seems harder.
  - Given data set S, finding apx best conjunction is NP-hard.
  - Can do other things, like minimize hinge-loss, but may be a big gap wrt error rate ("0/1 loss").
- One way to make progress: make assumptions on the "noise" in the data. E.g., Random Classification Noise model.

Learning from Random Classification Noise

- PAC model, target $f \in C$, but assume labels from noisy channel.
- "noisy" oracle $EX^{\eta}(f, D)$. η is the noise rate. (think $\eta = \frac{1}{4}$)
  - Example x is drawn from D.
  - With probability 1-η see label $\ell(x) = f(x)$.
  - With probability η see label $\ell(x) = 1 - f(x)$.
- E.g., if h has non-noisy error p, what is the noisy error rate? (If $\Pr_D[h(x) \neq f(x)] = p$, what is $\Pr_D[h(x) \neq \ell(x)]$?)
  - $p(1-\eta) + (1-p)\eta = \eta + p(1-2\eta)$.

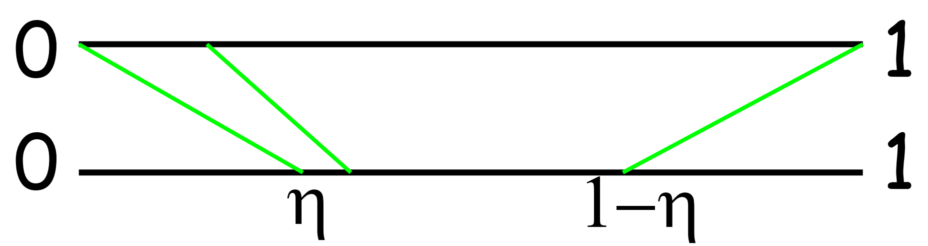


Learning from Random Classification Noise

Algorithm A PAC-learns C from random classification noise if for any f∈C, any distrib D, any η < 1/2, any ε, δ > 0, given access to $EX^{\eta}(f,D)$, A finds a hyp h that is ε-close to f, with probability $\geq$ 1-δ.

Want time poly(1/ε, 1/δ, 1/(1-2η), n , size(f))

- Q: is this a plausible goal? We are asking the learner to get closer to f than the data is.
- A: OK because noisy error rate is linear in true error rate (squashed by 1-2η)



Notation

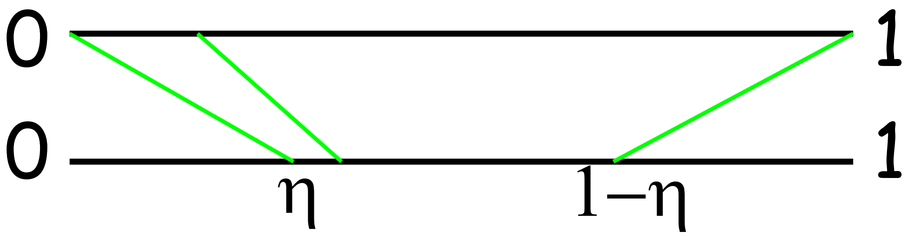
- Use "Pr[...]" for probability with respect to non-noisy distribution.
- Use "Pr_η[...]" for probability with respect to noisy distribution.

Learning OR-functions (assume monotone)

- Let's assume noise rate η is known.
- Say p_i = Pr[f(x)=0 and x_i=1] (if x_i in target then $p_i = 0$)
- Any h that includes all x_i such that p_i=0 and no x_i such that $p_i > \varepsilon/n$ is good. (e.g., think of $f = x_1 \vee x_3 \vee x_5$)
- So, just need to estimate p_i to $\pm\frac{\epsilon}{2n}$.
  - Rewrite as p_i = Pr[f(x)=0 | x_i=1] × Pr[x_i=1].
  - 2^{nd} part unaffected by noise (and if tiny, then p_i is small for sure). Define q_i as 1^{st} part.
  - Then $Pr_\eta[\ell(x)=0 | x_i=1] = q_i(1-\eta)+(1-q_i)\eta = \eta+q_i(1-2\eta)$.
  - So, enough to approx LHS to $\pm O\left(\frac{\epsilon}{2n}(1-2\eta)\right)$.

Learning OR-functions (assume monotone)

- If noise rate not known, can estimate with smallest value of $\Pr_\eta[\ell(x)=0 | x_i=1]$.



(e.g., $f = x_1 \vee x_3 \vee x_5$)

Generalizing the algorithm

Basic idea of algorithm was:

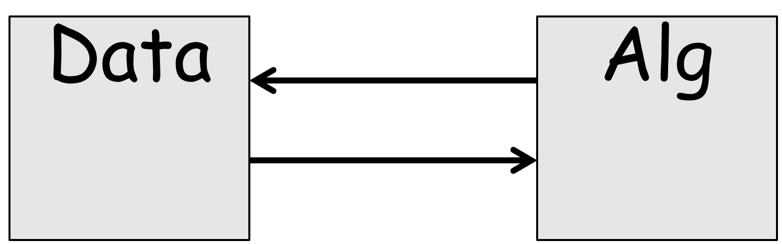
- See how can learn in non-noisy model by asking about probabilities of certain events with some "slop".
- Try to learn in noisy model by breaking events into:
  - Parts predictably affected by noise.
  - Parts unaffected by noise.

Let's formalize this in notion of "statistical query" (SQ) algorithm. Will see how to convert any SQ alg to work with noise.

The Statistical Query Model

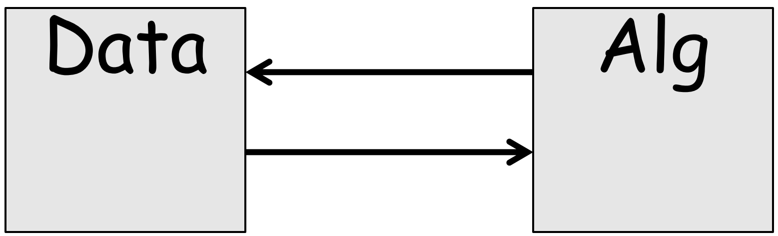
- No noise.
- Algorithm asks: "what is the probability a labeled example will have property χ? Please tell me up to additive error τ." (e.g., $x_i = 1$ and label is negative)
  - Formally, $\chi: X \times \{0,1\} \rightarrow \{0,1\}$. Must be poly-time computable. $\tau \geq 1/\text{poly}(\ldots)$.
  - Let $P_\chi = \Pr_{x \sim D}[\chi(x,f(x))=1]$.
  - World responds with $P'_\chi \in [P_\chi - \tau, P_\chi + \tau]$.

  [can extend to $E[\chi]$ for [0,1]-valued or vector-valued χ]
- May repeat poly(…) times. Can also ask for unlabeled data. Must output h of error $\leq \varepsilon$. No δ in this model.



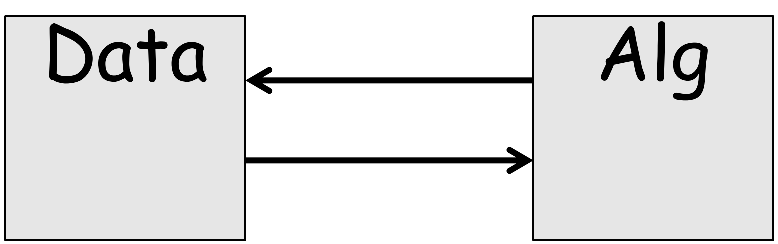
The Statistical Query Model

- Examples of queries:
  - What is the probability that x_i=1 and label is negative?
  - What is the error rate of my current hypothesis h? [$\chi(x,\ell)$=1 iff $h(x) \neq \ell$]
- Get back answer to $\pm\tau$. Can simulate from $\approx 1/\tau^2$ examples. [That's why need $\tau \geq 1/\text{poly}(\ldots)$.]
- To learn OR-functions, ask for Pr[x_i=1 and f(x)=0] with $\tau = \frac{\epsilon}{2n}$. Produce OR of all x_i s.t. $P'_\chi \leq \frac{\epsilon}{2n}$.



The Statistical Query Model

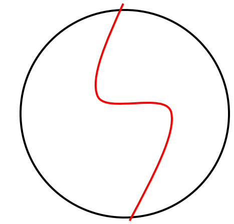
- Many algorithms can be simulated with statistical queries:
  - Perceptron: ask for $E[f(x)x : h(x) \neq f(x)]$ (formally define vector-valued $\chi = f(x)x$ if $h(x) \neq f(x)$, and 0 otherwise. Then divide by $Pr[h(x) \neq f(x)]$.)
  - Hill-climbing type algorithms: what is error rate of h? What would it be if I made this tweak?
- Properties of SQ model:
  - Can automatically convert to work in presence of classification noise.
  - Can give a nice characterization of what can and cannot be learned in it.



SQ-learnable ⇒ (PAC+Noise)-learnable

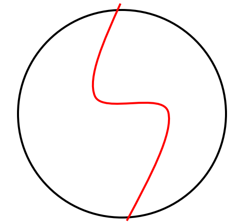
- Given query χ, need to estimate from noisy data. Idea:
  - Break into part predictably affected by noise, and part unaffected.
  - Estimate these parts separately.
  - Can draw fresh examples for each query or estimate many queries from same sample if VCDim of query space is small.
- Running example: $\chi(x,\ell)=1$ iff $x_i=1$ and $\ell=0$.

How to estimate $\Pr[\chi(x,f(x))=1]$?



- Let CLEAN = $\{x : \chi(x,0) = \chi(x,1)\}$
- Let NOISY = $\{x : \chi(x,0) \neq \chi(x,1)\}$
  - What are these for "$\chi(x,\ell)=1$ iff $x_i=1$ and $\ell=0$" ?
- Now we can write:
  - $\Pr[\chi(x,f(x))=1] = \Pr[\chi(x,f(x))=1 \text{ and } x \in \text{CLEAN}] +$
    $\Pr[\chi(x,f(x))=1 \text{ and } x \in \text{NOISY}]$.
- Step 1: first part is easy to estimate from noisy data (easy to tell if $x \in$ CLEAN).
- What about the 2nd part?

How to estimate $\Pr[\chi(x,f(x))=1]$?



- Let CLEAN = $\{x : \chi(x,0) = \chi(x,1)\}$
- Let NOISY = $\{x : \chi(x,0) \neq \chi(x,1)\}$
  - What are these for "$\chi(x,\ell)=1$ iff $x_i=1$ and $\ell=0$" ?
- Now we can write:
  - $\Pr[\chi(x,f(x))=1] = \Pr[\chi(x,f(x))=1 \text{ and } x \in \text{CLEAN}] + \Pr[\chi(x,f(x))=1 \text{ and } x \in \text{NOISY}]$.

- Can estimate $\Pr[x \in \text{NOISY}]$.
- Also estimate $P_\eta \equiv \Pr_\eta[\chi(x,\ell)=1 \mid x \in \text{NOISY}]$.
- Want $P \equiv \Pr[\chi(x,f(x))=1 \mid x \in \text{NOISY}]$.
- Write $P_\eta = P(1-\eta) + (1-P)\eta = \eta + P(1-2\eta)$.
- So, $P = (P_\eta - \eta)/(1-2\eta)$.
  - Just need to estimate P_η to additive error $\tau(1-2\eta)$.
  - If don't know η, can have "guess and check" wrapper.

So, any SQ algorithm can automatically be simulated in the presence of random classification noise

Characterizing what's learnable using SQ algorithms

- Say that f,g uncorrelated if $\Pr_{x\sim D}[f(x) = g(x)] = \frac{1}{2}$.

Def: the SQ-dimension of a class C wrt D is the size of the largest set $C' \subseteq C$ s.t. for all $f, g \in C'$,

$$\left|\Pr_D[f(x) = g(x)] - \frac{1}{2}\right| < \frac{1}{|C'|}.$$

(size of largest set of nearly uncorrelated functions in C)

- Theorem 1: if $SQDIM_D(C)$ = poly(n) then you can weak-learn C over D by SQ algs. [error rate $\leq \frac{1}{2} - \frac{1}{poly(n)}$]
- Theorem 2: if $SQDIM_D(C)$ > poly(n) then you can't weak-learn C over D by SQ algs.

Characterizing what's learnable using SQ algorithms

- Key tool: Fourier analysis of boolean functions.
- Sounds scary but it's a cool idea!
- Let's think of functions from $\{0,1\}^n \rightarrow \{-1,1\}$.
- View function f as a vector of 2^n entries:

$$\left(\sqrt{D[000]}f(000), \sqrt{D[001]}f(001), \ldots, \sqrt{D[x]}f(x), \ldots\right)$$

- What is $\langle f, f \rangle$? What is $\langle f, g \rangle$?
- What is an orthonormal basis?