TTIC 31250
An Introduction to the Theory of Machine Learning

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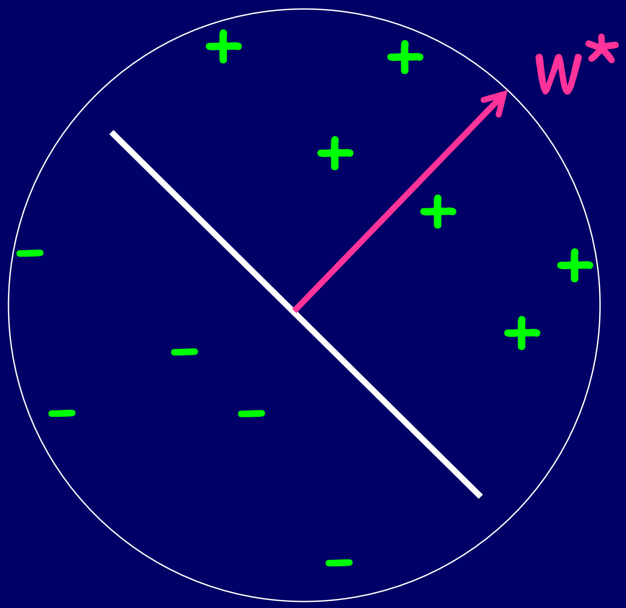
Lecture 3: The Perceptron Algorithm

Perceptron algorithm

Algorithm for learning a "large margin" linear separator in $\mathbb{R}^d$.

Online setting:

- Examples arrive one at a time.
- Given x, predict label y.
- Told correct answer.



Goal: bound number of mistakes under assumption there exists w^* such that $w^* \cdot x \geq 1$ on positives and $w^* \cdot x \leq -1$ on negatives.

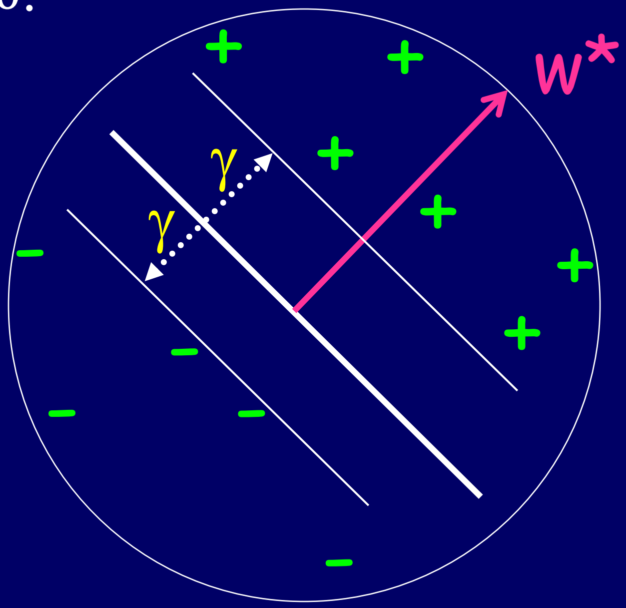
Perceptron alg: makes at most $\|w^*\|^2 \max(\|x\|^2)$ mistakes.

Perceptron algorithm

Perceptron alg makes $\leq \|w^*\|^2 \max(\ \|x\|^2\ )$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives.

How to think about this:

- $\frac{w^* \cdot x}{\|w^*\|}$ is distance of x to hyperplane $w^* \cdot x = 0$.
- Our assumption is equivalent to assuming exists a separator of margin $\gamma = \frac{1}{\|w^*\|}$.
- If points all lie in a ball of radius R, then mistake bound is at most R^2/γ^2.
- Notice this is scale-invariant.

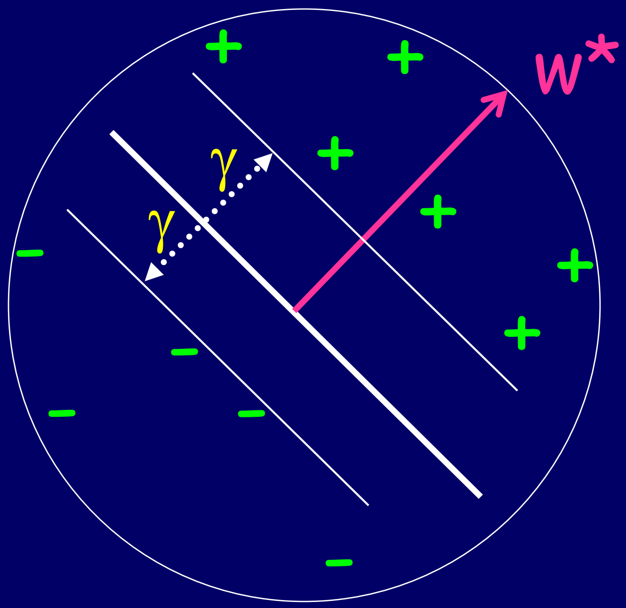


Perceptron algorithm

Perceptron alg makes $\leq \|w^*\|^2 \max(\,\|x\|^2\,)$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives.

Algorithm:

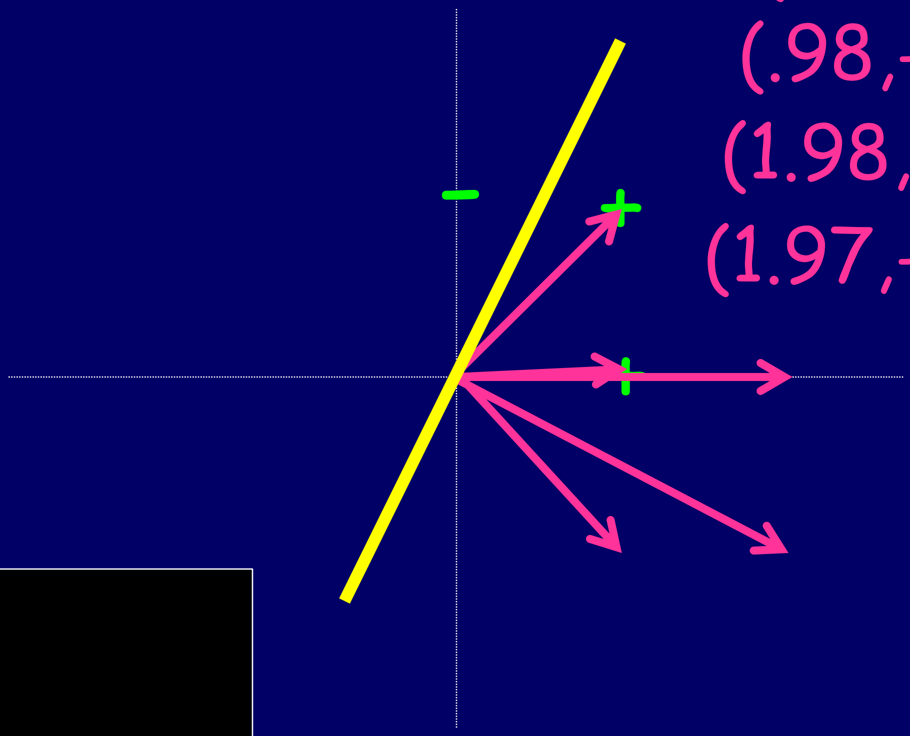
- Initialize $w = \vec{0}$. Predict positive if $w \cdot x > 0$, else predict negative.
- Mistake on positive: $w \leftarrow w + x$.
- Mistake on negative: $w \leftarrow w - x$.



Perceptron algorithm

Example:

(0.01,1) −
(1,1) +
(1,0) +
(0.01,1) −
(1,1) +
(1,0) +



Algorithm:

Initialize $w = \vec{0}$. Use $w \cdot x > 0$.

- Mistake on pos: $w \leftarrow w+x$.
- Mistake on neg: $w \leftarrow w-x$.

Analysis

Perceptron alg makes at most $\|w^*\|^2 R^2$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives, and all $\|x\| \leq R$.

Proof: consider w · w* and ||w||

- Each mistake increases w · w* by at least 1.

  (w + x) · w* = w · w* + x · w* $\geq$ w · w* + 1.

  So after M mistakes, $w \cdot w^* \geq M$.

- Each mistake increases w·w by at most R^2.

  (w + x) · (w + x) = w·w + 2(w·x) + x·x $\leq$ w·w + R^2.

  So, after M mistakes, $\|w\|^2 \leq MR^2$, so $\|w\| \leq \sqrt{M}R$.

Since $\frac{w \cdot w^*}{\|w^*\|} \leq \|w\|$, get $\frac{M}{\|w^*\|} \leq \sqrt{M}R$ so $\sqrt{M} \leq \|w^*\|R$.

Lower bound

Perceptron alg makes at most $\|w^*\|^2 R^2$ mistakes if $\exists w^*$ with $w^* \cdot x \geq 1$ on all positives and $w^* \cdot x \leq -1$ on all negatives, and all $\|x\| \leq R$.

In general it's not possible to get $< R^2/\gamma^2$ mistakes with a deterministic algorithm.

Proof: consider R^2/γ^2 coordinate vectors scaled to length R.

$$w^* = (\pm x_1 \pm x_2 \pm \cdots \pm x_{R^2/\gamma^2})/R$$

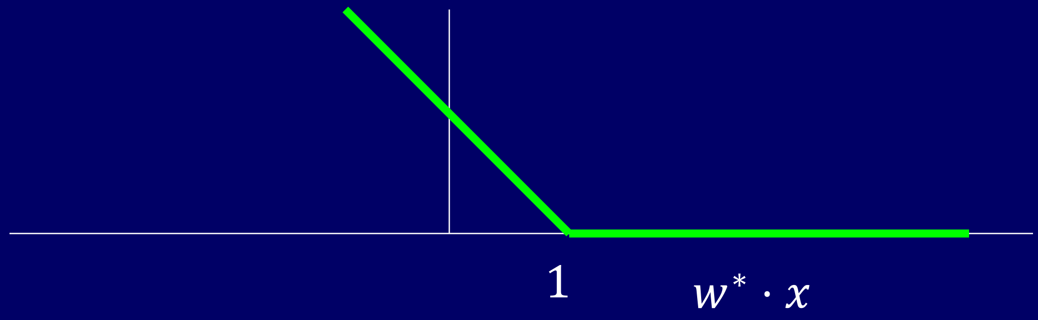
$|w^* \cdot x| = 1$ for all the input vectors, so can force all mistakes.

$\|w^*\| = \frac{\sqrt{R^2/\gamma^2}}{R} = \frac{1}{\gamma}$, so all margins are γ as desired.

What if no perfect separator?

In this case, a mistake could cause |w · w*| to drop.

The hinge-loss of w* on positive x is $\max(0, 1 - w^* \cdot x)$: the amount by which the inequality $w^* \cdot x \geq 1$ is not satisfied.



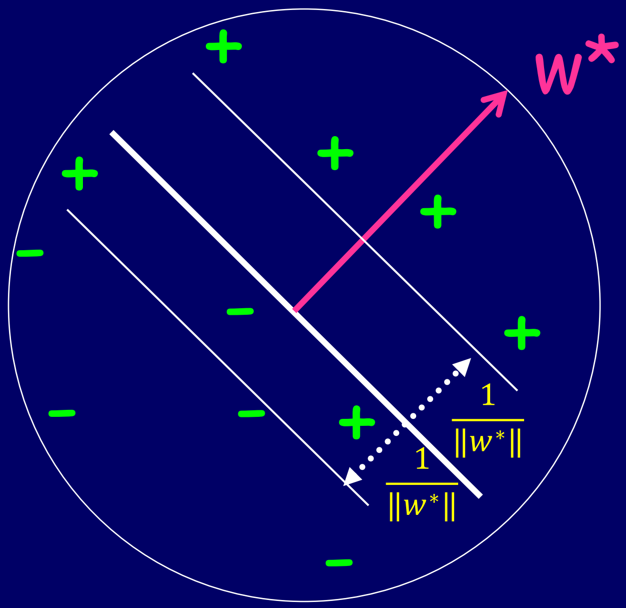
The hinge-loss of w* on negative x is $\max(0, 1 + w^* \cdot x)$: the amount by which the inequality $w^* \cdot x \leq -1$ is not satisfied.

What if no perfect separator?

In this case, a mistake could cause |w · w*| to drop.

Theorem: on any sequence of examples S, the Perceptron algo makes at most $\min_{w^*}\left[\|w^*\|^2 R^2 + 2L_{hinge}(w^*, S)\right]$ mistakes.

$L_{hinge}(w^*, S) =$ total hinge loss of w^* on set S.



Equivalently: how far you would have to move all the points to have them on the correct side by γ, in units of γ.

What if no perfect separator?

In this case, a mistake could cause |w · w*| to drop.

Theorem: on any sequence of examples S, the Perceptron algo makes at most $\min_{w^*}\left[\|w^*\|^2 R^2 + 2L_{hinge}(w^*, S)\right]$ mistakes.

Proof:

- After M mistakes, $w \cdot w^* \geq M - L_{hinge}(w^*, S)$.
- Still have: after M mistakes, $\|w\|^2 \leq MR^2$.
- Again use fact that $(w \cdot w^*)^2 \leq \|w\|^2 \|w^*\|^2$.
- Solve: $\left(M - L_{hinge}\right)^2 \leq MR^2\|w^*\|^2$. Do some algebra.

$$M^2 - 2ML_{hinge} + L_{hinge}^2 \leq MR^2\|w^*\|^2$$

$$M \leq R^2\|w^*\|^2 + 2L_{hinge} - L_{hinge}^2/M.$$

Finding large-margin separators

Can also modify Perceptron to find a large-margin separator.

The Margin Perceptron Algorithm(γ):

1. For convenience, assume all examples are normalized to have Euclidean length 1. Initialize $\mathbf{w}_1 = \ell(\mathbf{x})\mathbf{x}$, where $\mathbf{x}$ is the first example seen and initialize t to 1.
2. Predict positive if $\frac{\mathbf{w}_t \cdot \mathbf{x}}{||\mathbf{w}_t||} \geq \gamma/2$, predict negative if $\frac{\mathbf{w}_t \cdot \mathbf{x}}{||\mathbf{w}_t||} \leq -\gamma/2$, and consider an example to be a margin mistake when $\frac{\mathbf{w}_t \cdot x}{||\mathbf{w}_t||} \in (-\gamma/2, \gamma/2)$.
3. On a mistake (incorrect prediction or margin mistake), update as in the standard Perceptron algorithm: $\mathbf{w}_{t+1} \leftarrow \mathbf{w}_t + \ell(\mathbf{x})\mathbf{x}$; $t \leftarrow t + 1$.

Theorem 1 *Let $\mathcal{S}$ be a sequence of labeled examples consistent with a linear threshold function $\mathbf{w}^* \cdot \mathbf{x} > 0$, where $\mathbf{w}^*$ is a unit-length vector, and let*

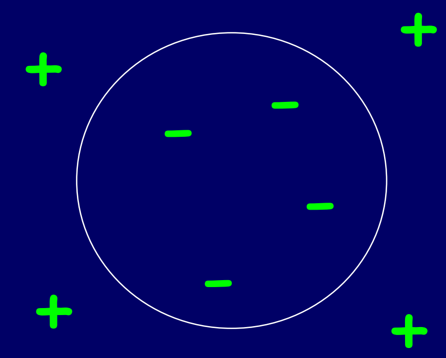
$$\gamma = \min_{\mathbf{x} \in \mathcal{S}} \frac{|\mathbf{w}^* \cdot \mathbf{x}|}{||\mathbf{x}||}.$$

Then the number of mistakes (including margin mistakes) made by Margin Perceptron(γ) on $\mathcal{S}$ is at most $8/\gamma^2$.

[BB06] M.F. Balcan and A. Blum. On a theory of learning with similarity functions. In *Proc. International Conference on Machine Learning (ICML)*, pages 73–80, 2006.

Kernel functions

What if the decision boundary between positive and negatives (e.g., spam and non-spam email) looks more like a circle than a linear separator?

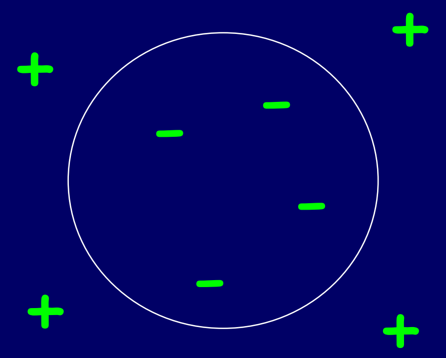


Idea: Kernel functions / "kernel trick":

- A pairwise function $K(x, x')$ is a kernel if there exists a function ϕ from input space to a new space (of possibly much higher dimension) such that $K(x, x') = \phi(x) \cdot \phi(x')$.
- Example: $K(x, x') = (1 + x \cdot x')^2$.
- Verify this is a kernel for special case that examples in $\mathbb{R}^2$:
- $K(x, x') = (1 + x_1 x_1' + x_2 x_2')^2 = 1 + 2x_1 x_1' + 2x_2 x_2' + x_1^2 x_1'^2 + 2x_1 x_2 x_1' x_2' + x_2^2 x_2'^2 = \phi(x) \cdot \phi(x')$ for $\phi(x) = (1, \sqrt{2}x_1, \sqrt{2}\, x_2, x_1^2, \sqrt{2}\, x_1 x_2, x_2^2)$.

Kernel functions

What if the decision boundary between positive and negatives (e.g., spam and non-spam email) looks more like a circle than a linear separator?



Idea: Kernel functions / "kernel trick":

- If can modify Perceptron so that only interacts with data via taking dot-products, and then replace $x \cdot x'$ with $K(x, x')$, then algorithm will act as if data was in higher-dimensional ϕ-space.
- Called "kernelizing" the algorithm.
- E.g., for $\phi(x) = \left(1, \sqrt{2}x_1, \sqrt{2}\, x_2, x_1^2, \sqrt{2}\, x_1 x_2, x_2^2\right)$, the weight vector $w^* = (-100, 0, 0, 1, 0, 1)$ gives a circle of radius 10 as decision boundary $w^* \cdot \phi(x) = 0$.

Kernel functions

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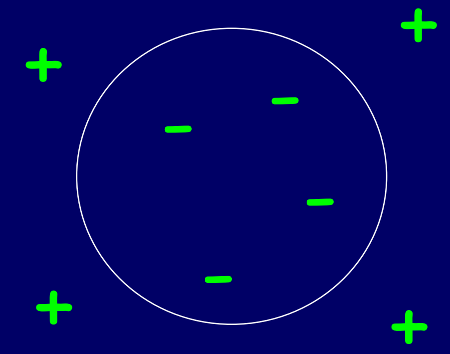

Idea: Kernel functions / "kernel trick":

- If can modify Perceptron so that only interacts with data via taking dot-products, and then replace $x \cdot x'$ with $K(x, x')$, then algorithm will act as if data was in higher-dimensional ϕ-space.
- How to kernelize Perceptron?
- Easy: weight vector always a sum of previous examples (or their negations), e.g., $w = x^{(1)} + x^{(3)} - x^{(6)}$. So, to predict on new x, just compute $w \cdot x = x^{(1)} \cdot x + x^{(3)} \cdot x - x^{(6)} \cdot x$. Now replace dot-product with kernel.

"Memorization Kernel"

What about a function $K(x, x') = \begin{cases} 1 \text{ if } x = x' \\ 0 \text{ if } x \neq x' \end{cases}$

Is that a kernel?

Wouldn't this make all datasets linearly separable?

What does the Perceptron mistake bound look like here?

Admin

Homework 1 due Wed. Hwk 2 will be made available Thurs

No class this coming Wed

Class on Monday next week will start a bit late (2:00)

References

[BB06] M.F. Balcan and A. Blum. On a theory of learning with similarity functions. In *Proc. International Conference on Machine Learning (ICML)*, pages 73–80, 2006.

[Blo62] H.D. Block. The perceptron: A model for brain functioning. *Reviews of Modern Physics*, 34:123–135, 1962. Reprinted in *Neurocomputing*, Anderson and Rosenfeld.

[FS99] Y. Freund and R.E. Schapire. Large margin classification using the Perceptron algorithm. *Machine Learning*, 37(3):277–296, 1999.

[MP69] M. Minsky and S. Papert. *Perceptrons: An Introduction to Computational Geometry*. The MIT Press, 1969.

[Nov62] A.B.J. Novikoff. On convergence proofs on perceptrons. In *Proceedings of the Symposium on the Mathematical Theory of Automata, Vol. XII*, pages 615–622, 1962.

[SSS05] S. Shalev-Shwartz and Y. Singer. A new perspective on an old perceptron algorithm. In *Proc. 16thh Annual Conference on Learning Theory*, 2005.

[TST05] P. Tsampouka and J. Shawe-Taylor. Analysis of generic perceptron-like large margin classifiers. In *ECML*, 2005.