TTIC 31260 - Algorithmic Game Theory (Winter 2026)

Homework # 4 **Solutions**

Exercises:

1. **Bilateral Trade.** Use the characterization of IC direct-revelation mechansisms to show that there is no IC direct revelation mechanism for the bilateral trade problem with property that:

   1. A trade happens if $v_b > v_s$ and not if $v_b < v_s$.
   2. The trade occurs at a price in $[v_s, v_b]$ with no subsidy by the mechanism.

   Solution: Consider, for concreteness, $v_b = 2$ and $v_s = 1$. In this case, a trade will have to occur at some price $p \in [1, 2]$. Suppose $p \neq 1$. Now consider an alternative buyer with $v_b' = (1 + p)/2$. For such a buyer, a trade would still have to happen (by condition 1) but from the characterization of IC mechanisms, the price charged to the buyer has to be the same as before (since both cases have the same outcome), so this violates condition (2). On the other hand, suppose $p = 1$. Then consider an alternative seller with $v_s' = 1.5$. Now we again get the same violation.

2. **Public Projects.** Use the characterization of IC direct-revelation mechansisms to show that there is no IC direct revelation mechanism for the problem of the government deciding whether to embark on a public project of cost C in a world with two citizens, with the property that: such that:

   1. If the two citizens' values on the project v_1 and v_2 satisfy $v_1 + v_2 > C$ then the project is undertaken and if $v_1 + v_2 < C$ then it's not.
   2. If the project is undertaken, then the citizens are in total charged at least C (i.e., no subsidy by the government), and citizen i is never charged more than v_i.

   Solution:

   Consider $v_1 = v_2 = 3C/4$. The price charged to player 1 cannot be a function of v_1, and it can't be larger than $C/4 + 1$ else the mechanism wouldn't act correctly for $v_1 = C/4 + 1$ (either it would fail to purchase or it would charge more than v_1). Similarly the price charged to player 2 can't be larger than $C/4 + 1$. So this violates the no-subsidy requirement.

3. **Revenue and reserve prices.** Consider an auction with one item and two bidders. Recall that a 2nd-price auction sells the item to the highest bidder at a price equal to the second-highest bid. A 2nd-price auction with reserve price p does the following: if the highest bid is $\geq p$ then it sells the item to the highest bidder at the maximum of the second-highest bid and p; else it does not sell the item at all.

Consider running a 2nd-price auction (with or without a reserve price) to two bidders whose valuations are known to be drawn independently at random from the uniform distribution on $[0, 1]$.

(a) What is the expected revenue of a second-price auction without a reserve price?

(b) What is the expected revenue of a second-price auction with reserve price $p = 1/2$?

You may use the fact that for independent random variables X and Y drawn uniformly from $[a, b]$, we have $E[\min(X, Y)] = a + (b - a)/3$.

Solution: For (a), the expected revenue is $E[\min(v_1, v_2)] = 1/3$.

For (b), we can calculate expected revenue as: $p^2 \cdot 0 + 2p(1-p)p + (1-p)^2(p + (1-p)/3) = 1/4 + (1/4)(2/3) = 5/12$.

Problems:

4. **Bidding languages.** Consider combinatorial auctions with m items. A valuation v is XOS if there exists a set of non-negative weight vectors $w_1, w_2, \ldots, w_t \in \mathbb{R}^m_{\geq 0}$ such that $v(S) = \max_j w_j \cdot x_S$ where $x_S \in \{0, 1\}^m$ is the indicator vector for set S. Show that $\{\text{submodular valuations}\} \subseteq \{\text{XOS valuations}\} \subseteq \{\text{subadditive valuations}\}$. In other words, any submodular valuation can be written as an XOS valuation, and any XOS valuation is subadditive. The first "$\subseteq$" relation is the hard one here.

   Hint: For thie first "$\subseteq$" relation, consider all possible permutations of the m items, and think about defining a weight vector for each of them.

   Solution:

   $\{$submodular valuations$\} \subseteq \{$XOS valuations$\}$: Let v be submodular and consider all permutations $\pi_1, \pi_2, \ldots$ of the m items. For each permutation π_j define the weight vector w_j where $w_{j,i}$ is the marginal value of the ith item in permutation π_j given all preceding items. I.e., $w_{j,i} = v(\{\pi_{j,1}, \ldots, \pi_{j,i}\}) - v(\{\pi_{j,1}, \ldots, \pi_{j,i-1}\})$. Define $v'(S) = \max_j w_j \cdot x_S$. Then the claim is that $v' = v$. First of all, it is clear that $v'(S) \geq v(S)$ since if we consider any permutation π_j in which the items in S all appear first, we have $w_j \cdot x_S = v(S)$ (by telescoping, the sum of the marginal values gives the value of the entire set). But now we use submodularity to get that for any other permutation π_k, we have $w_k \cdot x_S \leq v(S)$. In particular, consider permutation π_j that is identical to π_k in its ordering of the items in S, except all gaps are removed (all items not in S are moved to later). Then this can only increase the sum of the marginal values, due to the submodularity property that the marginal value of some new item i can only increase if you remove items from the current shopping cart. So, $w_j \cdot x_S \geq w_k \cdot x_S$.

   $\{$XOS valuations$\} \subseteq \{$subadditive valuations$\}$: Assume v is XOS and consider two sets S and T. Say $v(S \cup T)$ is given by $w \cdot x_{S \cup T}$. Then $v(S) \geq w \cdot x_S$ and $v(T) \geq w \cdot x_T$ (since each is a max, so they could be determined by w or by some other weight vector in v). So, $v(S) + v(T) \geq w \cdot x_S + w \cdot x_T \geq w \cdot x_{S \cup T}$ where the last inequality uses the fact that w has all entries non-negative.