TTIC 31260 - Algorithmic Game Theory (Winter 2026)

Homework # 3 **Solutions**

Exercises:

1. **Voting axioms.** Prove that any voting rule satisfying Condorcet Consistency must also satisfy Majority Consistency.

   Solution: If a majority of voters rank some candidate x first, then x will be a Condorcet winner, since it will beat every other candidate y in a pairwise election. So if a voting rule satisfies Condorcet Consistency, then it must select x whenever a majority of voters rank x first, meaning that it must satisfy Majority Consistency.

2. **Voting axioms II.** Arrow's impossibility theorem states that no social welfare function can satisfy all three of *unanimity*, *irrelevance of independent alternatives*, and *non-dictatorship*. Show that it is possible to achieve any two of the three.

   Solution: For Unanimity and IIA, you can just use a dictatorship that selects voter 1's ranking. For unanimity and non-dictatorship, there are a number of options; for example, you can rank candidates by their Borda Count votes: this is a non-dictatorship and has the property that if everyone ranks x above y, then x will get more points than y and so will be above y in the final ranking. Finally, for IIA and non-dictatorship, you can just always output some fixed ranking.

3. **Incentive-compatibility.** Consider selling two identical printers by collecting bids and then giving one printer to the highest bidder at price equal to the second-highest bid, and giving the other printer to the second-highest bidder at price equal to the third-highest bid.

   Show that this mechanism is *not* incentive-compatible by giving an explicit set of valuations v_1, v_2, v_3 (assume three bidders) such that at least one bidder i would receive higher utility by misreporting some $v_i' \neq v_i$.

   Solution: Consider $v_1 = 1, v_2 = 2, v_3 = 3$. If bidder 3 reports v_3 then they will receive the printer at a price of \$2. But if they misreport $v_3' = 1.5$ then they will receive the printer at a price of \$1. So, misreporting produces higher utility.

4. **VCG example.** Consider running the VCG mechanism with the Clarke pivot rule (i.e., the mechanism chooses the social-welfare-maximizing solution and each player is charged their externality) in the case of selling k identical printers. Specifically, assume each agent i has value $v_i \geq 0$ on obtaining a printer and no additional value for receiving more than one printer. For concreteness, let's also assume all v_i are distinct.

   Who gets the printers and what are they charged? Explain.

   Solution: The k highest bidders get the printers since this is the social welfare maximizing solution.

In terms of what each bidder is charged, recall that each bidder i is charged their externality, which is $\max_{a \in A} \left[\sum_{j \neq i} v_j(a)\right] - \sum_{j \neq i} v_j(f(v))$.

Let's denote the bids in sorted order as $v_1 \geq v_2 \geq \ldots \geq v_n$. Then for $i \leq k$ (i.e., bidder i gets a printer) we have $((v_1 + \ldots + v_{k+1}) - v_i) - ((v_1 + \ldots + v_k) - v_i) = v_{k+1}$. So, bidder i is charged v_{k+1}. For $i > k$ (i.e., bidder i does not get a printer), we have $(v_1 + \ldots + v_k) - (v_1 + \ldots + v_k) = 0$. So, bidder i is charged 0.

5. **VCG mechanism for a simple combinatorial auction.** Consider running a VCG mechanism with the Clarke pivot rule to auction two items, specifically a chair and a table. Assume we have three bidders who submit the following bids:

   - Bidder 1: \$10 for the chair, \$20 for the table, or \$25 for both together
   - Bidder 2: \$15 for the chair, \$15 for the table, or \$20 for both together
   - Bidder 3: \$10 for the chair, \$15 for the table, or \$30 for both together

   To whom will VCG allocate the items and how much will they be charged?

   Solution: VCG selects the social-welfare maximizing allocation, which is to give the table to bidder 1 and the chair to bidder 2.

   Bidder 1 is charged their externality, which is $\max_{a \in A} \left[\sum_{j \neq 1} v_j(a)\right] - \sum_{j \neq 1} v_j(f(v)) = 30 - 15 = 15$.

   Bidder 2 is charged their externality, which is $\max_{a \in A} \left[\sum_{j \neq 2} v_j(a)\right] - \sum_{j \neq 2} v_j(f(v)) = 30 - 20 = 10$.

   Bidder 3 is charged 0.