TTIC 31260 - Algorithmic Game Theory (Winter 2026)

Homework # 2 **Solutions**

Exercises:

1. **External regret vs Swap regret.**

   Consider playing T games of Rock-Paper-Scissors against an opponent who first plays Rock $T/3$ times, then plays Scissors $T/3$ times, then plays Paper $T/3$ times. Describe a sequence of (pure or mixed) strategies that would have zero external regret but $\Omega(T)$ swap-regret (and explain). Describe a second sequence of (pure or mixed) strategies with the same total payoff but that has no swap regret.

   Solution: For concreteness, let's say a win is worth 1, a tie is 0, and a loss is -1. Now, suppose you exactly mimic your opponent (this is a sequence of pure strategies). This will tie every time and so produce a total payoff of 0. This has no external regret since playing any fixed action will also have a total payoff of 0 since it wins $T/3$ times, loses $T/3$ times, and ties $T/3$ times. On the other hand, had you played Paper in the times you played Rock, played Scissors in the times you played Paper, and played Rock in the times you played Scissors, you could have won every time getting a payoff of T. So the swap regret is T.

   A second sequence of pure strategies with the same total payoff but with no swap regret is to play Rock every time. This has zero swap regret since the only options are to switch to a different fixed action, which also has total payoff of 0.

2. **Exercise 17.1 in the book.** The exact formula is not so enlightening, so instead of solving for the exact formula, argue what happens to the price of anarchy in the limit as $d \to \infty$.

   Solution: We are given that the top edge has cost 1 and the bottom edge has cost x^d when a fraction $x \in [0, 1]$ of players are using it. As with the original Pigou's example, at Nash equilibrium all players experience a cost of 1. To get intuition about the optimal solution, notice that if a $(1 - \frac{k}{d})$ of players use the bottom edge, then those players will incur cost $(1 - \frac{k}{d})^d \approx e^{-k}$. So, the overall total cost is $\frac{k}{d} + e^{-k}$. We can see that for any $k \in \omega(1) \cap o(d)$, such as $k = \ln d$, this cost goes to 0 as $d \to \infty$. So, the price of anarchy goes to infinity as $d \to \infty$.

3. **Exercise 17.2 in the book.**

   Solution: Now the price of stability is 1. Every player can go to v and then to s_k and then to t, for a total cost of $1/k$ split among all k players. This is the social optimum and also a Nash equilibrium.

Problems:

4. **Every Exact Potential Game is a Congestion Game.** In class, we showed that every congestion game is an exact potential game. Here you will show the converse (proven by Monderer and Shapley) that every exact potential game is also a congestion game.

   Specifically, assume you are given a game G with n players and exact potential function Φ. For convenience, you may assume that each player has the same number of action choices L. Your job is to define a congestion game such that in every state $s = (s_1, ..., s_n)$, the costs to each player are identical to their costs in that state in G. We will do this in two stages:

   (a) First, define a congestion game where in every state $s = (s_1, ..., s_n)$, each player incurs cost $\Phi(s)$. (See hints below)

   (b) Now modify this game so that each player has the correct cost according to game G. (See hints below)

   Hints: define 2^{nL} resources, where each resource is an $n \times L$ matrix of bits. Then define action j for player i as choosing all resources with a 1 in entry (i, j). What you now need to do is define cost functions for each resource, first to achieve goal (a) and then to achieve goal (b).

   To aid in this, for each state s, let R_s^{min} be the minimal resource (the resource with the fewest 1's in it) such that all players have that resource in state s. What does R_s^{min} look like? Solve part (a) by only creating cost functions for these resources $\{R_s^{min}\}_{s \in S}$ (all other resources have cost of 0).

   Now, to solve part (b), let $R_{s,i}^{max}$ be the maximal resource (the resource with the most 1's in it) such that only player i has that resource in state s. What does $R_{s,i}^{max}$ look like? Solve part (b) by taking your solution to part (a) and then also creating cost functions for these $R_{s,i}^{max}$ resources too. Note: these cost functions need not be monotone in the number of users. You should argue why your solution works and is well-defined.

   Solution: For part (a), define resources and the notion of R_s^{min} as in the hint. For state $s = (s_1, ..., s_n)$, resource R_s^{min} has 1's in entries (i, s_i) for each $i \in \{1, ..., n\}$ and 0's in all other entries. We define the cost of that resource R_s^{min} to be the function $f_{R_s^{min}}(k) = \Phi(s)$ if $k = n$ and $f_{R_s^{min}}(k) = 0$ if $k \neq n$. All other resources have cost 0. Notice that in any state s, each player pays cost $\Phi(s)$ for using resource R_s^{min}, and pays cost 0 for all the other resources it is using, because all resources $R_{s'}^{min}$ for $s' \neq s$ are being used by strictly fewer than n players.

   For part (b), define $R_{s,i}^{max}$ as in the hint. This resource is all 1's except with 0's in entries $(i', s_{i'})$ for all $i' \neq i$. Notice that this resource is always used by player i, and has exactly one player using it iff either (a) we are in state s or (b) we are in some state $s' = (s_i', s_{-i})$ for $s_i' \neq s_i$. We will give this resource a cost of 0 if more than one player is using it, and a cost of $cost_i(s) - \Phi(s)$ is exactly one player (namely, player i) is using it. We have to be careful here: notice that $R_{s,i}^{max} = R_{s',i}^{max}$ for all $s' = (s_i', s_{-i})$ so we had better make sure this is well-defined; that is, we need to check that $cost_i(s) - \Phi(s) = cost_i(s') - \Phi(s')$ for all such s'. Luckily, this equation holds *by definition of an exact potential game. In fact, this is the one place we use the fact that the game is an exact potential game.* So, we do this, and now for any player i, for any state s, the cost of player i in state s is exactly $\Phi(s) + cost_i(s) - \Phi(s) = cost_i(s)$ as desired.