

TTIC 31260 - Algorithmic Game Theory (Winter 2026)

Homework # 1

Due: January 14, 2026

Groundrules:

- You should work by yourself on the *exercises* but may work with a partner on the *problems* if you want. (Working together doesn't mean "splitting up the problems" though.) If you work with a partner, then write down who you are working with.
- If you've seen a problem before (sometimes I'll give problems that are "famous"), then say that in your solution. It won't affect your score, I just want to know. Also, if you use any sources other than the AGT book, write that down too. It's fine to look up a complicated sum or inequality, but please don't look up an entire solution.

Exercises:

1. **Go with your strengths?** Recall that in the penalty-shot game the shooter can shoot left or shoot right, and the goalie can dive left or dive right. If the goalie guesses correctly then the shot is blocked, else the ball goes in and it is a goal. Now, suppose the shooter is wildly inaccurate on the left, so that even if the goalie dives right, the ball still has only a 25% chance of going in. I.e., the payoff matrix R for the shooter (the row player) looks like:

		Goalie	
		left	right
Shooter	left	0	1/4
	right	1	0

(The payoff matrix C for the goalie equals $-R$).

- (a) What is the minimax optimal strategy for the shooter, and what is the minimax optimal value?
 - (b) Is there anything perhaps counterintuitive about how weakening the shooter on the left affected its minimax optimal strategy?
 - (c) What is the minimax optimal strategy for the goalie?
2. **Exchangability.** Suppose that (p, q) and (p', q') are both Nash equilibria for some game given by payoff matrices (R, C) .
 - (a) Must (p, q') be a Nash equilibrium? If so, give a proof; if not, give a counterexample.

(b) Suppose the game is zero-sum ($C = -R$). Must (p, q') be a Nash equilibrium? If so, give a proof; if not, give a counterexample.

3. **Game-theoretic RWM (multiplicative weights) analysis.** In class we described the RWM algorithm in the context of predicting from expert advice, and briefly sketched how the analysis extended to the game-theoretic setting where the algorithm is *selecting an action* rather than combining predictions. Here we want you to go ahead and do that analysis.

Specifically, assume you have N action choices (rows in a matrix game). The algorithm maintains a weight w_i for each row i (initialized to $w_i = 1 \forall i$) and defines probability distribution $p_i = w_i/W$ where $W = \sum_i w_i$. To make this a little different from the discussion in class, let's assume that payoffs are *gains* in the range $[0, 1]$ and that if action i receives a gain of g_i then the algorithm updates w_i using

$$w_i \leftarrow w_i(1 + \epsilon g_i),$$

where $0 < \epsilon \leq 1$ is an input to the algorithm.

Let $G_i = \sum_t g_i^t$ denote the cumulative gain of action i over all time steps t so far (superscript t denotes the time-step), and let $Q_i = \sum_t (g_i^t)^2 \leq G_i$. Let G_{ALG} denote the expected cumulative gain of the above algorithm. Prove that

$$G_{ALG} \geq \max_i [G_i - \epsilon Q_i/2 - (\ln N)/\epsilon] \geq \max_i [G_i(1 - \epsilon/2) - (\ln N)/\epsilon].$$

Feel free to look at the proof of the closely-related Theorem 4.6 in the book if you like. Also you will want to use the fact that for $x \in [0, 1]$ we have $x - x^2/2 \leq \ln(1 + x) \leq x$.

Problems:

4. **On approximate Nash equilibria.** Consider a two-player n -by- n general-sum game. Recall that in a Nash equilibrium (p, q) , for each i s.t. $p_i > 0$ we have $e_i^T Rq = \max_{i'} e_i^T Rq$ and similarly for each j s.t. $q_j > 0$ we have $p^T C e_j = \max_{j'} p^T C e_{j'}$.

Now, assume we have a game in which all payoffs are in the range $[0, 1]$. Define a pair of distributions (p, q) to be an ϵ -Nash equilibrium if each player has at most ϵ incentive to deviate. Even more stringently, define a pair of distributions (p, q) to be a *well-supported* ϵ -Nash equilibrium if each row i with $p_i > 0$ satisfies $e_i^T Rq \geq \max_{i'} e_i^T Rq - \epsilon$ and each row j with $q_j > 0$ satisfies $p^T C e_j \geq \max_{j'} p^T C e_{j'} - \epsilon$.¹

Using the fact that Nash equilibria must exist, show that there must exist an ϵ -Nash equilibrium (in fact, a well-supported ϵ -Nash) in which each player has positive probability on at most $O(\frac{1}{\epsilon^2} \log n)$ actions (rows or columns).

¹So the difference is that in an ϵ -Nash equilibrium, players can put a small amount of probability on very bad actions, whereas in a well-supported ϵ -Nash they cannot.

Hint: you will want somewhere to use Hoeffding's inequality, which says that if $X = \frac{1}{m} \sum_{i=1}^m X_i$ where the X_i are independent $[0, 1]$ -valued random variables, then

$$\Pr[|X - \mathbf{E}[X]| > \epsilon] \leq 2e^{-2m\epsilon^2}.$$

Note: this fact yields an $n^{O(\frac{1}{\epsilon^2} \log n)}$ -time algorithm for finding an ϵ -Nash equilibrium. No PTAS (algorithm running in time polynomial in n for any fixed $\epsilon > 0$) is known, however.

5. **Tracking a moving target.** Here is a variation on the deterministic Weighted-Majority algorithm, designed to make it more adaptive.
- (a) Each expert begins with weight 1 (as before).
 - (b) We predict the result of a weighted-majority vote of the experts (as before).
 - (c) If an expert makes a mistake, we penalize it by dividing its weight by 2, but *only* if its weight was at least $1/4$ of the average weight of experts.

Prove that in any contiguous block of trials (e.g., the 51st example through the 77th example), the number of mistakes made by the algorithm is at most $O(m + \log N)$, where m is the number of mistakes made by the best expert *in that block*, and N is the total number of experts.