

01/28/26

Price of Anarchy, Price of Stability,
Potential & Congestion Games

Your guide:

Avrim Blum

[Readings: Ch. 17, 19.3 of AGT book]

High level

Now, switching to...

- Games with many players, but structured
 - Network routing, resource sharing,...
- Examining different questions
 - How much do we lose in terms of overall “quality” of the solution, if players are self-interested

General setup

- n players. Player i chooses strategy $s_i \in S_i$.
- Overall state $s = (s_1, \dots, s_n) \in S$.
[Will only be considering pure strategies]
 - Utility function $u_i: S \rightarrow \mathbb{R}$, or
 - Cost function $\text{cost}_i: S \rightarrow \mathbb{R}$.
 - **(Sum) Social Welfare** of s is sum of utilities over all players.
 - If costs, called **Sum Social Cost**.
 - Other things to care about: happiness of least-happy player, etc.

Price of Anarchy / Price of Stability

n players. Player i chooses strategy $s_i \in S_i$.

Say we're talking costs, so lower is better.

Price of Anarchy:

Ratio of cost of worst equilibrium to cost of social optimum. (worst-case over games in class)

Price of Stability:

Ratio of cost of best equilibrium to cost of social optimum. (worst-case over games in class)

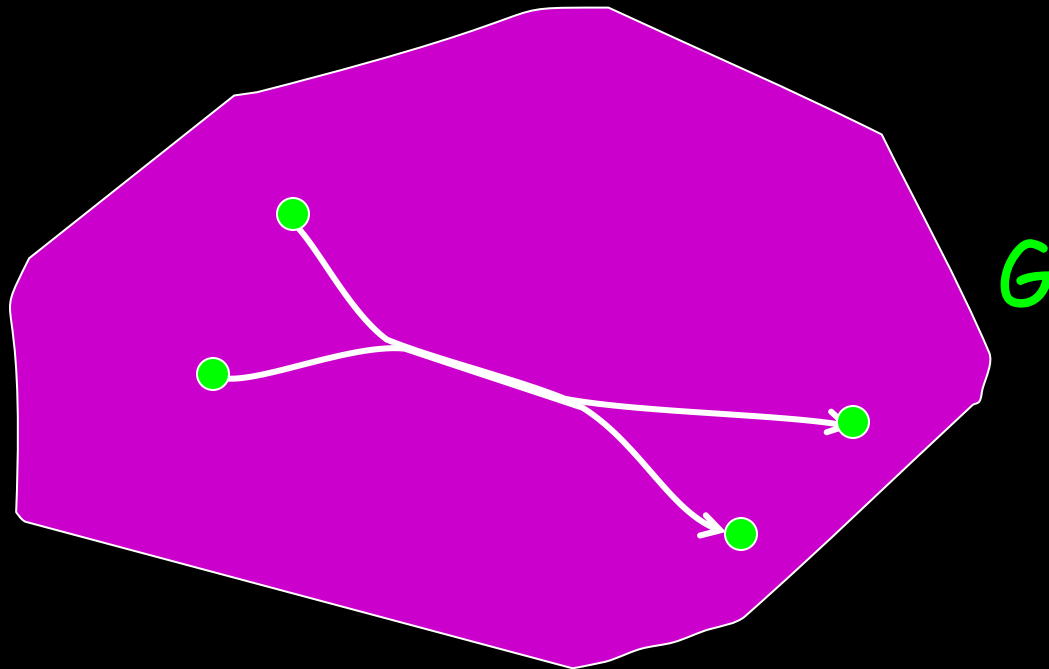
Example: Fair Cost-Sharing

- n players in weighted directed graph G .
- Player i wants to get from s_i to t_i .
- Each edge e has cost c_e .
- Players **share** the cost of edges they use with others using it.

Overloading s_i
here - sorry.

This is
what
makes it
a game

We will
care about
sum social
cost

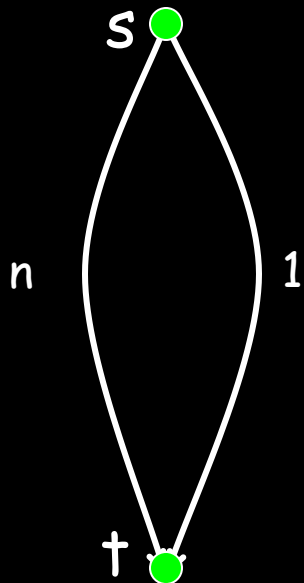


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Also equilib



Social optimum: all use edge of cost 1.
(cost $1/n$ per player; total = 1)

Bad equilibrium: all use edge of cost n .
(cost 1 per player; total = n)

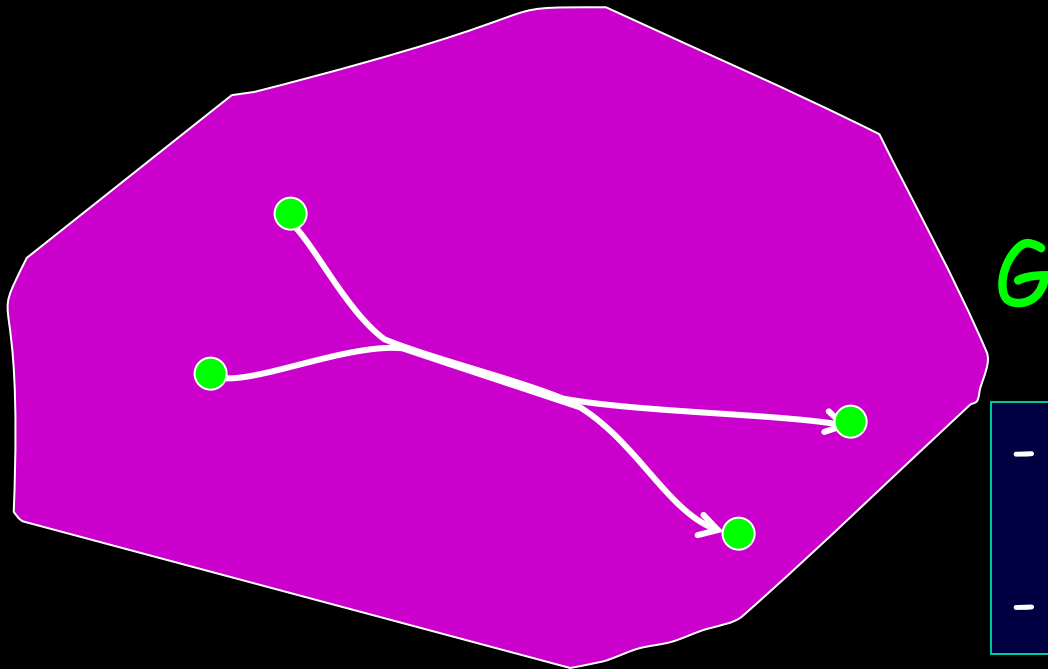
So, Price of Anarchy $\geq n$.

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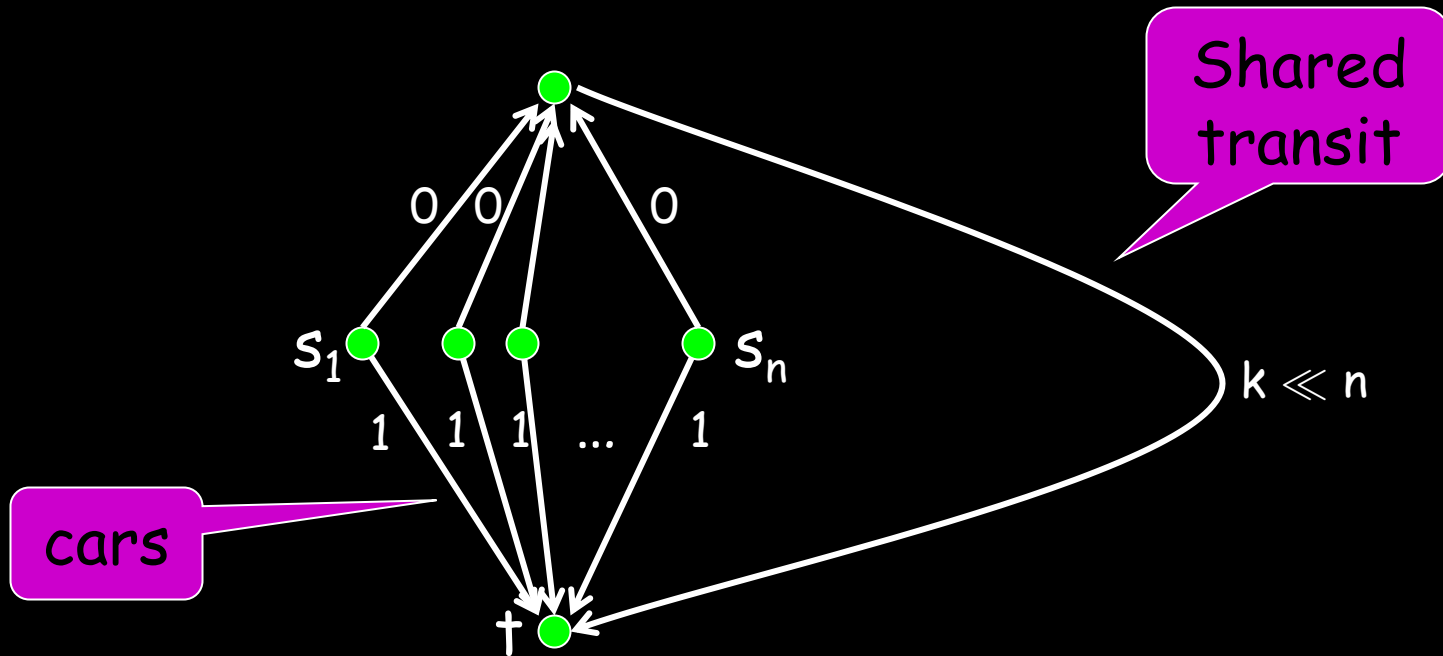
Can anyone see
argument that Price
of Anarchy $\leq n$?



- $\text{Cost}(\text{NE}) \leq \sum_i \text{SP}(s_i, t_i)$.
- $\text{Cost}(\text{OPT}) \geq \max_i \text{SP}(s_i, t_i)$.

Example: Fair Cost-Sharing

One more interesting example.



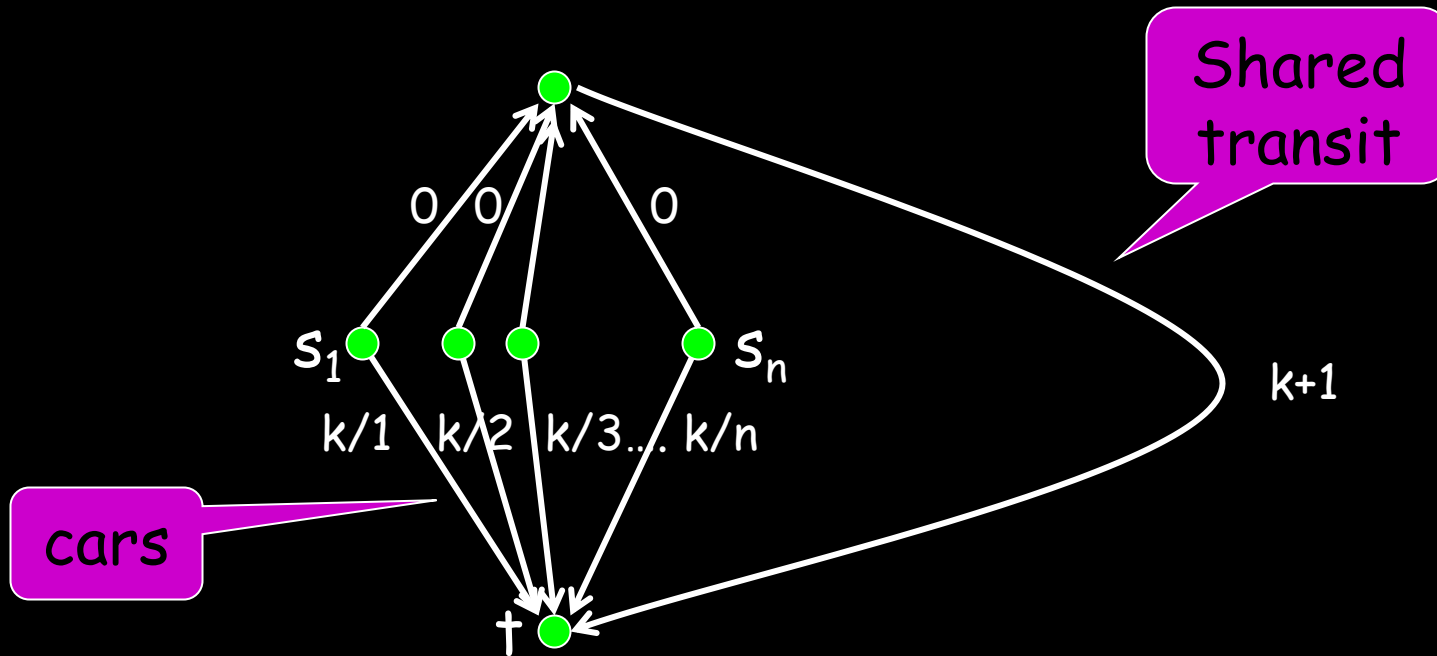
OPT has cost k (and is equilib). Also NE of cost n .

Now, let's modify it...

Example: Fair Cost-Sharing

One more interesting example.

Price of Stability
= $\Omega(\log n)$



OPT has cost $k+1$. Only equilib has cost $k \ln n$.

Now, let's modify it...

Example: Fair Cost-Sharing

In fact, Price of Stability for fair cost-sharing is $O(\log n)$ too.

For this, we will use the fact that fair cost-sharing is an **exact potential game**...

Exact Potential Games

G is an exact potential game if there exists a function $\Phi(s)$ such that:

- For all players i , for all states $s = (s_i, s_{-i})$, for all possible moves to state $s' = (s'_i, s_{-i})$,

$$\text{cost}_i(s') - \text{cost}_i(s) = \Phi(s') - \Phi(s)$$

- Notice that this implies there must exist a pure-strategy Nash equilibrium. Why?
- Furthermore, can reach by simple best-response dynamics. Each move is guaranteed to reduce the potential function.

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Claim: Fair cost-sharing is an exact potential game.

- Define potential $\Phi(s) = \sum_e \sum_{i=1}^{n_e(s)} c_e / i$
- If player changes from path p to path p' , pays $c_e / (n_e(s) + 1)$ for each new edge, gets back $c_e / n_e(s)$ for each old edge. So, $\Delta \text{cost}_i = \Delta \Phi$.

Interesting fact about this potential

What is the gap between potential and cost?

$$\text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s).$$

What does this imply about PoS?

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What is the gap between potential and cost?

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What does this imply about PoS?

- Say we start at socially optimal state OPT.
- Do best-response dynamics from there until reach Nash equilibrium s .
- $\text{cost}(s) \leq \Phi(s) \leq \Phi(\text{OPT}) \leq \log(n) \times \text{cost}(\text{OPT})$.

So, Price of Stability = $O(\log n)$.

Fair cost-sharing summary

In every game:

- \forall equilib s , $\text{cost}(s) \leq n \times \text{cost}(\text{OPT})$.
- \exists equilib s , $\text{cost}(s) \leq \log(n) \times \text{cost}(\text{OPT})$.

There exist games s.t.

- \exists equilib s , $\text{cost}(s) \geq n \times \text{cost}(\text{OPT})$.
- \forall equilib s , $\text{cost}(s) \geq \text{clog}(n) \times \text{cost}(\text{OPT})$.

Furthermore, potential function satisfies:

$$\text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s).$$

So, starting from an arbitrary state, people optimizing for themselves can hurt overall cost but not too much.

Congestion Games more generally

Game defined by n players and m resources.

- Each player i chooses a **set** of resources (e.g., a path) from collection S_i of allowable sets of resources (e.g., paths from s_i to t_i).
- Cost of resource j is a function $f_j(n_j)$ of the number n_j of players using it.
- Cost incurred by player i is the sum, over all resources they are using, of the cost of the resource.
- Generic potential function:
$$\sum_j \sum_{i=1}^{n_j} f_j(i)$$
- Best-response dynamics may take a long time to reach equilib, but if gap between Φ and cost is small, can get to apx-equilib fast by having plr with most incentive go next.

Congestion Games & Potential Games

We just saw that every congestion game is an exact potential game.

[Rosenthal '73]

Turns out the converse is true as well.

[Monderer and Shapley '96]

For any exact potential game, can define resources to view it as a congestion game.

[see hwk]