Exercises:

0. **Course Project.** One of the class requirements is to do a course project. This could be reading a recent paper related to class topics and explaining it in a 3-5 page writeup; it could be theoretically investigating a question related to class topics and writing up your thoughts in a 3-5 page report; or it could be conducting an experimental investigation and writing what you did and found in a 3-5 page report. Your project report is due on May 15. For this homework, your job is to think about it and write “I thought about it”.

1. **Voting axioms.** Prove that any voting rule satisfying Condorcet Consistency must also satisfy Majority Consistency.

   Solution: If a majority of voters rank some candidate $x$ first, then $x$ will be a Condorcet winner, since it will beat every other candidate $y$ in a pairwise election. So if a voting rule satisfies Condorcet Consistency, then it must select $x$ whenever a majority of voters rank $x$ first, meaning that it must satisfy Majority Consistency.

2. **Voting axioms II.** Arrow’s impossibility theorem states that no social welfare function can satisfy all three of **unanimity**, **irrelevance of independent alternatives**, and **non-dictatorship**. Show that it is possible to achieve any two of the three.

   Solution: For Unanimity and IIA, you can just use a dictatorship that selects voter 1’s ranking. For unanimity and non-dictatorship, there are a number of options; for example, you can rank candidates by their Borda Count votes: this is a non-dictatorship and has the property that if everyone ranks $x$ above $y$, then $x$ will get more points than $y$ and so will be above $y$ in the final ranking. Finally, for IIA and non-dictatorship, you can just always output some fixed ranking.

3. **Incentive-compatibility.** Consider selling two identical printers by collecting bids and then giving one printer to the highest bidder at price equal to the second-highest bid, and giving the other printer to the second-highest bidder at price equal to the third-highest bid.

   Show that this mechanism is not incentive-compatible by giving an explicit set of valuations $v_1, v_2, v_3$ (assume three bidders) such that at least one bidder $i$ would receive higher utility by misreporting some $v'_i \neq v_i$.

   Solution: Consider $v_1 = 1, v_2 = 2, v_3 = 3$. If bidder 3 reports $v_3$ then they will receive the printer at a price of $S2$. But if they misreport $v'_3 = 1.5$ then they will receive the printer at a price of $S1$. So, misreporting produces higher utility.
4. **VCG example.** Consider running the VCG mechanism with the Clarke pivot rule (i.e., the mechanism chooses the social-welfare-maximizing solution and each player is charged their externality) in the case of selling $k$ identical printers. Specifically, assume each agent $i$ has value $v_i \geq 0$ on obtaining a printer and no additional value for receiving more than one printer. For concreteness, let’s also assume all $v_i$ are distinct.

Who gets the printers and what are they charged? Explain.

**Solution:** The $k$ highest bidders get the printers since this is the social welfare maximizing solution.

In terms of what each bidder is charged, recall that each bidder $i$ is charged their externality, which is $\max_{a \in A} \left[ \sum_{j \neq i} v_j(a) - \sum_{j \neq i} v_j(f(v)) \right]$.

Let’s denote the bids in sorted order as $v_1 \geq v_2 \geq \ldots \geq v_n$. Then for $i \leq k$ (i.e., bidder $i$ gets a printer) we have $((v_1 + \ldots + v_{k+1}) - v_i) - ((v_1 + \ldots + v_k) - v_i) = v_{k+1}$. So, bidder $i$ is charged $v_{k+1}$. For $i > k$ (i.e., bidder $i$ does not get a printer), we have $(v_1 + \ldots + v_k) - (v_1 + \ldots + v_k) = 0$. So, bidder $i$ is charged 0.

5. **VCG mechanism for a simple combinatorial auction.** Consider running a VCG mechanism with the Clarke pivot rule to auction two items, specifically a chair and a table. Assume we have three bidders who submit the following bids:

- Bidder 1: $10 for the chair, $20 for the table, or $25 for both together
- Bidder 2: $15 for the chair, $15 for the table, or $20 for both together
- Bidder 3: $10 for the chair, $15 for the table, or $30 for both together

To whom will VCG allocate the items and how much will they be charged?

**Solution:** VCG selects the social-welfare maximizing allocation, which is to give the table to bidder 1 and the chair to bidder 2.

Bidder 1 is charged their externality, which is $\max_{a \in A} \left[ \sum_{j \neq 1} v_j(a) - \sum_{j \neq 1} v_j(f(v)) \right] = 30 - 15 = 15$.

Bidder 2 is charged their externality, which is $\max_{a \in A} \left[ \sum_{j \neq 2} v_j(a) - \sum_{j \neq 2} v_j(f(v)) \right] = 30 - 20 = 10$.

Bidder 3 is charged 0.