04/03/24

Price of Anarchy, Price of Stability, Potential & Congestion Games

Your guide:
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[Readings: Ch. 17, 19.3 of AGT book]
High level

Now, switching to…

• Games with many players, but structured
  – Network routing, resource sharing,…

• Examining different questions
  – How much do we lose in terms of overall “quality” of the solution, if players are self-interested
General setup

n players. Player i chooses strategy $s_i \in S_i$.

- Overall state $s = (s_1, \ldots, s_n) \in S$.
  
  [Will only be considering pure strategies]

- Utility function $u_i : S \rightarrow \mathbb{R}$, or

- Cost function $\text{cost}_i : S \rightarrow \mathbb{R}$.

- (Sum) Social Welfare of $s$ is sum of utilities over all players.

- If costs, called Sum Social Cost.

- Other things to care about: happiness of least-happy player, etc.
Price of Anarchy / Price of Stability

An n players. Player \( i \) chooses strategy \( s_i \in S_i \).

Say we’re talking costs, so lower is better.

**Price of Anarchy:**

Ratio of cost of worst equilibrium to cost of social optimum. *(worst-case over games in class)*

**Price of Stability:**

Ratio of cost of best equilibrium to cost of social optimum. *(worst-case over games in class)*
Example: Fair Cost-Sharing

- $n$ players in weighted directed graph $G$.
- Player $i$ wants to get from $s_i$ to $t_i$.
- Each edge $e$ has cost $c_e$.
- Players share the cost of edges they use with others using it.

This is what makes it a game.

We will care about sum social cost.

Overloading $s_i$ here - sorry.
Example: Fair Cost-Sharing

- $n$ players in weighted directed graph $G$.
- Player $i$ wants to get from $s_i$ to $t_i$.
- Each edge $e$ has cost $c_e$.
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Social optimum: all use edge of cost 1. (cost $1/n$ per player; total = 1)

Bad equilibrium: all use edge of cost $n$. (cost 1 per player; total = $n$)

So, Price of Anarchy $\geq n$. 

Overloading $s_i$ here - sorry.

Also equilib
Example: Fair Cost-Sharing

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- Player $i$ wants to get from $s_i$ to $t_i$.
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Can anyone see argument that Price of Anarchy $\leq n$?

\[ \text{Cost(NE)} \leq \sum_i \text{SP}(s_i, t_i). \]
\[ \text{Cost(OPT)} \geq \max_i \text{SP}(s_i, t_i). \]
Example: Fair Cost-Sharing

One more interesting example.

OPT has cost $k$ (and is equilib). Also NE of cost $n$.

Now, let’s modify it...
Example: Fair Cost-Sharing

One more interesting example.

OPT has cost $k+1$. Only equilib has cost $k \ln n$.

Now, let’s modify it...

Price of Stability $= \Omega(\log n)$

Shared transit
Example: Fair Cost-Sharing

In fact, Price of Stability for fair cost-sharing is $O(\log n)$ too.

For this, we will use the fact that fair cost-sharing is an exact potential game...
Exact Potential Games

$G$ is an exact potential game if there exists a function $\Phi(s)$ such that:

- For all players $i$, for all states $s = (s_i, s_{-i})$, for all possible moves to state $s' = (s_i', s_{-i})$,

\[ \text{cost}_i(s') - \text{cost}_i(s) = \Phi(s') - \Phi(s) \]

- Notice that this implies there must exist a pure-strategy Nash equilibrium. Why?

- Furthermore, can reach by simple best-response dynamics. Each move is guaranteed to reduce the potential function.
Exact Potential Games

$G$ is an exact potential game if there exists a function $\Phi(s)$ such that:

- For all players $i$, for all states $s = (s_i, s_{-i})$, for all possible moves to state $s' = (s'_i, s'_{-i})$,

$$\text{cost}_i(s') - \text{cost}_i(s) = \Phi(s') - \Phi(s)$$

Claim: Fair cost-sharing is an exact potential game.

- Define potential $\Phi(s) = \sum_{e} \sum_{i=1}^{n_e(s)} c_e / i$

- If player changes from path $p$ to path $p'$, pays $c_e / (n_e(s) + 1)$ for each new edge, gets back $c_e / n_e(s)$ for each old edge. So, $\Delta \text{cost}_i = \Delta \Phi$. 
Interesting fact about this potential

What is the gap between potential and cost?

$$\text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s).$$

What does this imply about PoS?

Claim: Fair cost-sharing is an exact potential game.

- Define potential $$\Phi(s) = \sum_{e} \sum_{i=1}^{n_{e}(s)} \frac{c_{e}}{i}$$
- If player changes from path $$p$$ to path $$p'$$, pays $$\frac{c_{e}}{n_{e}(s)+1}$$ for each new edge, gets back $$\frac{c_{e}}{n_{e}(s)}$$ for each old edge. So, $$\Delta \text{cost}_i = \Delta \Phi.$$
Interesting fact about this potential

What is the gap between potential and cost?

\[ \text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s). \]

What does this imply about PoS?

- Say we start at socially optimal state state OPT.
- Do best-response dynamics from there until reach Nash equilibrium \( s \).
- \[ \text{cost}(s) \leq \Phi(s) \leq \Phi(\text{OPT}) \leq \log(n) \times \text{cost}(\text{OPT}). \]

So, Price of Stability = \( O(\log n) \).
Fair cost-sharing summary

In every game:

- \( \forall \) equilibria \( s \), \( \text{cost}(s) \leq n \times \text{cost}(\text{OPT}) \).
- \( \exists \) equilibria \( s \), \( \text{cost}(s) \leq \log(n) \times \text{cost}(\text{OPT}) \).

There exist games s.t.

- \( \exists \) equilibria \( s \), \( \text{cost}(s) \geq n \times \text{cost}(\text{OPT}) \).
- \( \forall \) equilibria \( s \), \( \text{cost}(s) \geq \text{clog}(n) \times \text{cost}(\text{OPT}) \).

Furthermore, potential function satisfies:

\[ \text{cost}(s) \leq \Phi(s) \leq \log(n) \times \text{cost}(s). \]

So, starting from an arbitrary state, people optimizing for themselves can hurt overall cost but not too much.
Congestion Games more generally

Game defined by $n$ players and $m$ resources.

- Each player $i$ chooses a set of resources (e.g., a path) from collection $S_i$ of allowable sets of resources (e.g., paths from $s_i$ to $t_i$).

- Cost of resource $j$ is a function $f_j(n_j)$ of the number $n_j$ of players using it.

- Cost incurred by player $i$ is the sum, over all resources being used, of the cost of the resource.

- Generic potential function:
  $$\sum_j \sum_{i=1}^{n_j} f_j(i)$$

- Best-response dynamics may take a long time to reach equilib, but if gap between $\Phi$ and cost is small, can get to apx-equilib fast.
Congestion Games & Potential Games

We just saw that every congestion game is an exact potential game.

[Rosenthal '73]

Turns out the converse is true as well.

[Monderer and Shapley '96]

For any exact potential game, can define resources to view it as a congestion game.

[see hwk]
Note

Next class we’ll have a break 1:50-2:15 to go downstairs for the eclipse watch party.