Mathematical Toolkit	**Autumn 2021**

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1 Orthogonality and orthonormality

Definition 1.1 *Two vectors u, v in an inner product space are said to be* orthogonal *if $\langle u, v \rangle = 0$. A set of vectors $S \subseteq V$ is said to consist of* mutually orthogonal vectors *if $\langle u, v \rangle = 0$ for all $u \neq v$, $u, v \in S$. A set of $S \subseteq V$ is said to be* orthonormal *if $\langle u, v \rangle = 0$ for all $u \neq v$, $u, v \in S$ and $\|u\| = 1$ for all $u \in S$.*

Proposition 1.2 *A set $S \subseteq V \setminus \{0_V\}$ consisting of mutually orthogonal vectors is linearly independent.*

Proof: Let $v_1, \ldots, v_n \in S$ and $c_1, \ldots, c_n \in \mathbb{F}$ be such that $\sum_{i \in [n]} c_i \cdot v_i = 0_V$. Taking inner product with a vector v_j for $j \in [n]$, we get that $\sum_i c_i \cdot \langle v_j, v_i \rangle = 0$. Since vectors in S are mutually orthogonal, we get that $\langle v_j, v_i \rangle = 0$ when $i \neq j$, which implies using the previous equality that that $c_j \langle v_j, v_j \rangle = 0$. Since $v_j \neq 0_V$, we must have $\langle v_j, v_j \rangle > 0$, and thus $c_j = 0$. Also, since our choice of j was arbitrary, this is true for all $j \in [n]$, implying $c_1 = \cdots = c_n = 0$. Thus, the only way a finite linear combination of vectors from S equals 0_V, if all coefficients are 0 , which implies that S is linearly independent. ■

Proposition 1.3 (Gram-Schmidt orthogonalization) *Given a finite set $\{v_1, \ldots, v_n\}$ of linearly independent vectors, there exists a set of orthonormal vectors $\{w_1, \ldots, w_n\}$ such that*

$$\mathrm{Span}\left(\{w_1, \ldots, w_n\}\right) = \mathrm{Span}\left(\{v_1, \ldots, v_n\}\right).$$

Proof: By induction. The case with one vector is trivial. Given the statement for k vectors and orthonormal $\{w_1, \ldots, w_k\}$ such that

$$\mathrm{Span}\left(\{w_1, \ldots, w_k\}\right) = \mathrm{Span}\left(\{v_1, \ldots, v_k\}\right),$$

define

$$u_{k+1} = v_{k+1} - \sum_{i=1}^{k} \langle w_i, v_{k+1} \rangle \cdot w_i \quad \text{and} \quad w_{k+1} = \frac{u_{k+1}}{\|u_{k+1}\|}.$$

It is easy to check that the set $\{w_1, \ldots, w_{k+1}\}$ satisfies the required conditions. ■

Corollary 1.4 *Every finite dimensional inner product space has an orthonormal basis.*

In fact, Hilbert spaces also have orthonormal bases (which are countable). The existence of a maximal orthonormal set of vectors can be proved by using Zorn's lemma, similar to the proof of existence of a Hamel basis for a vector space. However, we still need to prove that a maximal orthonormal set is a basis. This follows because we define the basis slightly differently for a Hilbert space: instead of allowing only finite linear combinations, we allow infinite ones. The correct way of saying this is that is we still think of the span as the set of all *finite* linear combinations, then we only need that for any $v \in V$, we can get arbitrarily close to v using elements in the span (a converging sequence of finite sums can get arbitrarily close to its limit). Thus, we only need that the span is *dense* in the Hilbert space V. However, if the maximal orthonormal set is not dense, then it is possible to show that it cannot be maximal. Such a basis is known as a Hilbert basis.

1.1 Fourier coefficients

Let V be a finite dimensional inner product space and let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then for any $v \in V$, there exist $c_1, \ldots, c_n \in \mathbb{F}$ such that $v = \sum_i c_i \cdot w_i$. The coefficients c_i are often called Fourier coefficients (of v, with respect to the basis B). Using the orthonormality and the properties of the inner product, we get

Proposition 1.5 *Let $B = \{w_1, \ldots, w_n\}$ be an orthonormal basis for V, and let $v \in V$ be expressible as $v = \sum_{i=1}^{n} c_i \cdot w_i$. Then, for all $i \in [n]$, we must have $c_i = \langle w_i, v \rangle$.*

This can be used to prove the following

Proposition 1.6 (Parseval's identity) *Let V be a finite dimensional inner product space and let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then, for any $u, v \in V$*

$$\langle u, v \rangle = \sum_{i=1}^{n} \overline{\langle w_i, u \rangle} \cdot \langle w_i, v \rangle \, .$$

2 Adjoint of a linear transformation

Definition 2.1 *Let V, W be inner product spaces over the same field $\mathbb{F}$ and let $\varphi : V \to W$ be a linear transformation. A transformation $\varphi^* : W \to V$ is called an* adjoint *of φ if*

$$\langle w, \varphi(v) \rangle = \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W \, .$$

Example 2.2 *Let $V = W = \mathbb{C}^n$ with the inner product $\langle u, v \rangle = \sum_{i=1}^{n} u_i \cdot \overline{v_i}$. Let $\varphi : V \to V$ be represented by the matrix A. Then φ^* is represented by the matrix $\overline{A^T}$.*

Example 2.3 *Let $V = C([0,1],[-1,1])$ with the inner product defined as $\langle f_1, f_2 \rangle = \int_0^1 f_1(x) f_2(x) dx$, and let $W = C([0,1/2],[-1,1])$ with the inner product $\langle g_1, g_2 \rangle = \int_0^{1/2} g_1(x) g_2(x) dx$. Let $\varphi : V \to W$ be defined as $\varphi(f)(x) = f(2x)$. Then, $\varphi^* : W \to V$ can be defined as*

$$\varphi^*(g)(y) = (1/2) \cdot g(y/2) .$$

Exercise 2.4 *Let $\varphi_{\text{left}} : Fib \to Fib$ be the left shift operator as before, and let $\langle f, g \rangle$ for $f, g \in Fib$ be defined as $\langle f, g \rangle = \sum_{n=0}^{\infty} \frac{f(n)g(n)}{C^n}$ for $C > 4$. Find φ_{left}^*.*

We will prove that every linear transformation has a unique adjoint. However, we first need the following characterization of linear transformations from V to $\mathbb{F}$.

Proposition 2.5 (Riesz Representation Theorem) *Let V be a finite-dimensional inner product space over $\mathbb{F}$ and let $\alpha : V \to \mathbb{F}$ be a linear transformation. Then there exists a unique $z \in V$ such that $\alpha(v) = \langle z, v \rangle \ \forall v \in V$.*

We only prove the theorem here for finite-dimensional spaces. However, the theorem holds for any Hilbert space, as long as the linear transformation is "continuous"'.

Proof: Let $\{w_1, \ldots, w_n\}$ be an orthonormal basis for V. Then check that

$$z \;=\; \sum_{i=1}^{n} \overline{\alpha(w_i)} \cdot w_i$$

must be the unique z satisfying the required property. ∎

This can be used to prove the following:

Proposition 2.6 *Let V, W be finite dimensional inner product spaces and let $\varphi : V \to W$ be a linear transformation. Then there exists a unique $\varphi^* : W \to V$, such that*

$$\langle w, \varphi(v) \rangle \;=\; \langle \varphi^*(w), v \rangle \quad \forall v \in V, w \in W .$$

Proof: For each $w \in W$, the map $\langle w, \varphi(\cdot) \rangle : V \to \mathbb{F}$ is a linear transformation (check!) and hence there exists a unique $z_w \in V$ satisfying $\langle w, \varphi(v) \rangle = \langle z_w, v \rangle \ \forall v \in V$. Consider the map $\beta : W \to V$ defined as $\beta(w) = z_w$. By definition of β,

$$\langle w, \varphi(v) \rangle \;=\; \langle \beta(w), v \rangle \quad \forall v \in V, w \in W .$$

To check that α is linear, we note that $\forall v \in V, \ \forall w_1, w_2 \in W$,

$$\langle \beta(w_1 + w_2), v \rangle \;=\; \langle w_1 + w_2, \varphi(v) \rangle \;=\; \langle w_1, \varphi(v) \rangle + \langle w_2, \varphi(v) \rangle \;=\; \langle \beta(w_1), v \rangle + \langle \beta(w_2), v \rangle ,$$

which implies $\beta(w_1 + w_2) = \beta(w_1) + \beta(w_2)$ (why?) $\beta(c \cdot w) = c \cdot \beta(w)$ follows similarly. ∎

Note that the above proof only requires the Riesz representation theorem (to define z_w), and hence also works for Hilbert spaces (when φ is continuous).